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A Fallacy in Performance Measurement and Risk Analysis

1. Introduction

Performance measurement is based on geometric means. Therefore, one is tempted to use geometric means as well for the estimation of risk premia (expected excess returns relative to cash) on bonds and equities. Such a procedure may have rather counter-intuitive consequences. In section 2 we show by means of historical data that situations may occur, where diversification not only reduces risk but also increases returns. This phenomenon is explained in section 3 in the framework of a discrete time model. If the asset returns are independent, identically distributed (i.i.d.) for the different periods of time then geometric means systematically underestimate risk premia. As shown in section 4 this result holds as well in a continuous time model, where the price dynamics of assets are given by geometric Brownian motions.

2. A Puzzling Example

The performance of a Swiss equity portfolio from the end of 1987 to the end of 1998 was 20.58% p.a., measured by the MSCI Switzerland (total return index). CHF 100 invested at the beginning of the period grew to CHF 783 at the end of the period ($100 * 1.2058^{11} = 783$). Of course, a Swiss investor would not invest 100% of the portfolio in Swiss equities. So let's consider a portfolio of Swiss and US equities. A portfolio of US equities alone grew from USD 100 to USD 709 in the same period. In CHF terms, it grew from CHF 100 to CHF 763, or 20.29% p.a. Measured in the same currency, the performance of both markets was quite similar. What was the performance of a portfolio of 50% US equities and 50% Swiss equities, rebalanced yearly, measured in CHF?

A first guess of most readers is probably something between 20.29% and 20.58%, with an end-of-period value of something between CHF 763 and CHF 783 (at least this was the authors' guess or expectation). It turns out that the performance of this portfolio was 20.65% p.a., with a terminal value of CHF 789. In fact, the performance was better than performance of both markets alone! Diversification not only reduced risk, it also increased return.

In the next section we show that this phenomenon is due to the fact that building portfolio returns involves arithmetic averages, whereas building average returns over time involves geometric averages.

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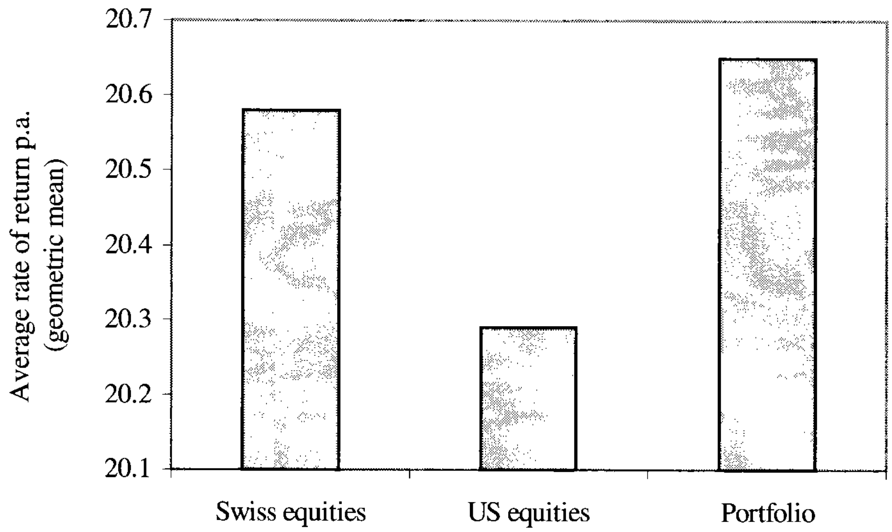
Figure 1: Annualized Rate of Return for Swiss Equities, US equities and a Portfolio of Swiss and US Equities, 1988–1998



3. Discrete Time Model

We assume that there are two assets with stochastic returns ($r_{1,t}$, $r_{2,t}$) for periods t = 1,..., T. The return of a portfolio (a, 1–a) is given by

$$r_{P,t} = a\, r_{1,t} + (1-a)\, r_{2,t} \tag{1}$$

$S_{1,t}$, $S_{2,t}$, $S_{P,t}$, t = 0, 1,..., T denote end of period prices of the assets, respectively of the portfolio.

First we look at the puzzle of section 2.
Taking logarithmic portfolio returns, we get

$$\begin{aligned} \ln\left(1+r_{P,t}\right) &= \ln\left(1+a\cdot r_{1,t}+(1-a)\cdot r_{2,t}\right) \\ &\geq a\cdot \ln\left(1+r_{1,t}\right)+(1-a)\cdot \ln\left(1+r_{2,t}\right) \end{aligned} \tag{2}$$

because the natural logarithm is a concave function. Compounding the one-period-returns between 0 and T leads to:

$$\ln\left(\frac{S_{i,T}}{S_{i,0}}\right) = \sum_{t=1}^{T} \ln\left(1+r_{i,t}\right) \qquad , i = 1, 2, P \tag{3}$$

As (2) holds in every period, (2) and (3) can be combined to yield

$$\begin{aligned} \ln\left(\frac{S_{P,T}}{S_{P,0}}\right) &\geq a\cdot \sum_{t=1}^{T}\ln\left(1+r_{1,t}\right)+(1-a)\cdot \sum_{t=1}^{T}\ln\left(1+r_{2,t}\right) \\ &= a\cdot \ln\left(\frac{S_{1,T}}{S_{1,0}}\right)+(1-a)\cdot \ln\left(\frac{S_{2,T}}{S_{2,0}}\right) \end{aligned} \tag{4}$$

The left-hand side is strictly greater if the return of the two stocks are unequal in at least one period (which is of course always true in practice).
From (4) one obtains

$$\left(\frac{S_{P,T}}{S_{P,0}}\right)^{\frac{1}{T}} \geq \left(\frac{S_{1,T}}{S_{1,0}}\right)^{\frac{a}{T}} \left(\frac{S_{2,T}}{S_{2,0}}\right)^{\frac{1-a}{T}} \tag{5}$$

For the special case

$$\frac{S_{1,T}}{S_{1,0}} = \frac{S_{2,T}}{S_{2,0}}$$

this leads to

$$\left(\frac{S_{P,T}}{S_{P,0}}\right)^{\frac{1}{T}} \geq \left(\frac{S_{1,T}}{S_{1,0}}\right)^{\frac{1}{T}} = \left(\frac{S_{2,T}}{S_{2,0}}\right)^{\frac{1}{T}} \tag{6}$$

This solves the puzzle of section 2. It shows that the portfolio's return per period is greater than the weighted average of the stock's returns per period. In the special case of equal total returns over the T periods for both assets, the portfolio return is greater than the return of the individual assets unless their returns are equal in each period. Note that we didn't make any assumption regarding the distribution of stock prices. The conclusion only depends on the assumption that stock returns are not equal in every period.
Next we show that in the case where $(r_{1,t}, r_{2,t})$, $t = 1,\ldots,T$ are independent, identically distributed (i.i.d.) geometric means systematically underestimate returns also for $T \to \infty$.
Denote

$$E\left[1+r_{i,t}\right] = v_i \quad , E\left[\ln\left(1+r_{i,t}\right)\right] = u_i \quad , i = 1, 2 \tag{7}$$

Hence v_i denotes the expected return which is relevant for portfolio analysis.
The i.i.d. assumption implies

$$E\left(\frac{S_{i,T}}{S_{i,0}}\right) = \prod_{t=1}^{T} E\left(1+r_{i,t}\right) = v_i^T \tag{8}$$

and therefore

$$v_i = \left[E\left(\frac{S_{i,T}}{S_{i,0}}\right)\right]^{\frac{1}{T}} \tag{9}$$

Due to (3) the geometric mean can be written as

$$\left(\frac{S_{i,T}}{S_{i,0}}\right)^{\frac{1}{T}} = e^{\frac{1}{T}\sum_{t=1}^{T}\ln\left(1+r_{i,t}\right)} \tag{10}$$

Applying the strong law of large numbers on the exponent of the right hand side shows the almost sure convergence of the geometric mean estimator

$$\left(\frac{S_{i,T}}{S_{i,0}}\right)^{\frac{1}{T}} \xrightarrow[T\to\infty]{} e^{u_i} \tag{11}$$

However, JENSEN'S inequality yields

$$\begin{aligned} e^{u_i} &= e^{E\left[\ln\left(1+r_{i,t}\right)\right]} < E\left[e^{\ln\left(1+r_{i,t}\right)}\right] \\ &= E\left[1+r_{i,t}\right] = v_i \end{aligned} \tag{12}$$

Hence the limit of the geometric mean estimator e^{u_i} which is appropriate for performance measurement is systematically lower than the expected return v_i which should be used for MARKOWITZ portfolio optimization.

4. Continuous Time Model

The question remains to which extent the puzzle of section 2 is a consequence of a "too slow" rebalancing. What happens to the effect if the portfolio is rebalanced more frequently? After all, the returns in equation (2) get smaller when the time period gets shorter, and one may ask what happens in the limit. To answer this question, and in order to analyze the size of this "diversification effect", we formulate the problem in a continuous time setting. In fact, in a continuous time setting the phenomenon becomes even more obvious and can be explained by standard ITÔ calculus.
We use the standard Brownian motion model for stock prices:

$$\frac{dS_i}{S_i} = \mu_i \cdot dt + \sigma_i \cdot dW_i \quad , i = 1, 2 \tag{13}$$

The portfolio is continuously rebalanced according to the weights a and (1 – a) and can therefore be described by the following equation:

$$\frac{dS_P}{S_P} = (a\mu_1 + (1-a)\mu_2)\cdot dt + a\cdot\sigma_1\cdot dW_1 + (1-a)\cdot\sigma_2\cdot dW_2 = (a\mu_1 + (1-a)\mu_2)\cdot dt + \sigma_P\cdot dW_P \quad (14)$$

Using ITÔ'S lemma, we get the following expression for the natural log's of the end-of-period values[1]:

$$\ln(S_{i,T}) = \ln(S_{i,0}) + \int_0^T \left(\mu_i - \frac{1}{2}\sigma_i^2\right)\cdot dt + \int_0^T \sigma_i\cdot dW_{i,t} \quad, i = 1, 2, P \quad (15)$$

In this formulation, the drift terms, which represent the expected continuously compounded rates of return, differ from the drift in (13) by 0.5 times the variance of the respective stock or portfolio.

First of all, it becomes obvious from (15) that the geometric mean is a consistent estimator of $e^{\mu_i - \frac{\sigma_i^2}{2}}$ rather than of e^{μ_i} (see also section 3).

Secondly, formula (15) allows us to explain the puzzle of section 2 in a continuous time setting. If the two stocks are not perfectly correlated, the downward correction for the portfolio is relatively small, compared with the downward corrections of the stocks. More precisely, the portfolio variance is smaller than the weighted average of stock variances if ρ < 1. Define y as half of this difference:

$$y \equiv \frac{1}{2}\cdot\left[a\cdot\sigma_1^2 + (1-a)\cdot\sigma_2^2 - \sigma_P^2\right] = \frac{1}{2}\cdot\left[\begin{array}{l} a\cdot\sigma_1^2 + (1-a)\cdot\sigma_2^2 \\ -\left(\begin{array}{l} a^2\cdot\sigma_1^2 + 2\cdot a\cdot(1-a)\cdot\rho\,\sigma_1\sigma_2 \\ +(1-a)^2\,\sigma_2^2 \end{array}\right)\end{array}\right] = \frac{1}{2}\left[a\cdot(1-a)\cdot\left(\sigma_1^2 + \sigma_2^2 - 2\cdot\rho\,\sigma_1\sigma_2\right)\right] > 0 \quad (16)$$

As an immediate consequence, the expected continuously compounded rate of return of the portfolio – the drift term – is greater than the weighted average of the expected continuously compounded rates of return of the two stocks. In the special case of equal expected continuously compounded rates of returns for both stocks, forming a portfolio leads to a greater expected continuously compounded rate of return.

Looking at realized returns instead of expected returns is even more interesting. Note that the portfolio's stochastic term is the weighted average of the two stock's stochastic terms due to the continuous rebalancing of the portfolio:

$$\int_0^T \sigma_P\cdot dW_{P,t} = a\cdot\int_0^T \sigma_1\cdot dW_{1,t} + (1-a)\cdot\int_0^T \sigma_2\cdot dW_{2,t} \quad (17)$$

This leads to the following relationship between end-of-period values of the two stocks and the portfolio:

$$\ln\left(\frac{S_{P,T}}{S_{P,0}}\right) = \int_0^T \left(\mu_P - \frac{1}{2}\sigma_P^2\right)\cdot dt + \int_0^T \sigma_P\cdot dW_{P,t} = \int_0^T \left(\begin{array}{l} a\cdot\mu_1 + (1-a)\cdot\mu_2 \\ -\frac{1}{2}\cdot\left(a\cdot\sigma_1^2 + (1-a)\cdot\sigma_2^2\right) - y\end{array}\right)\cdot dt + a\cdot\int_0^T \sigma_1\cdot dW_{1,t} + (1-a)\cdot\int_0^T \sigma_2\cdot dW_{2,t} = a\cdot\ln\left(\frac{S_{1,t}}{S_{1,0}}\right) + (1-a)\cdot\ln\left(\frac{S_{2,t}}{S_{2,0}}\right) + yT \quad (18)$$

It follows that if an equal amount is invested in the portfolio and in each of the stocks and if by chance the returns of the two stocks over the period under consideration are equal, the end-of-period portfolio value is greater than the end-of-period values of the single stock portfolios. It is also possible to formulate this relationship in terms of ex post continuously compounded average rate of returns between 0 and T:

$$\overline{r_P} = a \cdot \overline{r_1} + (1-a) \cdot \overline{r_2} + y \tag{19}$$

The definition of y (see (16)) shows that the effect is greatest for the equally weighted portfolio (a = 1/2) because it depends on the drift correction term (subtraction of 0.5 times the variance from the original drift term) which enters into the picture when we analyze the natural log's of the stock and portfolio values.
The correction term for the portfolio is smaller than the weighted averages of the stock's correction terms because the portfolio variance is reduced due to diversification. The smaller the correlation between the two stocks, the higher the ex post portfolio return. Hence, diversification truly reduces risk and due to an upward bias in measured ex post returns it seemingly increases returns.

5. Conclusion

In a discrete and in a continuous framework it was shown that the geometric mean is a downward biased estimator for expected returns. Portfolio diversification leads to a decrease of this bias. For ex post considerations (performance measurement) geometric means are used and diversification seems to increase returns. To avoid confusion ex post returns should be exclusively used for performance measurement. For risk return analysis unbiased return estimators are needed.
Coming back to the example mentioned at the beginning, if arithmetic means are used for expected returns estimation[2], we get 22.60% for Swiss equities, 22.20% for US equities and 22.40% for the portfolio - in line with our intuition.

Footnotes

[1] HULL (1997), pp. 220–221, or OKSENDAHL (1998), p.62.

[2] This calculation is done for illustrative purposes. In sophisticated portfolio optimization reverse optimization techniques are used instead of direct expected returns estimations based on data over a few years.

References

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