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Duration and Convexity for Bond Portfolios

1. Introduction

Duration as a measure of the interest rate risk of a bond has been given great attention in the literature, see e.g. BIERWAG, KAUFMAN and TOEVS (1983) for a survey, but much less attention has been given to the duration of bond portfolios. However, BIERWAG, CORRADO and KAUFMAN (1990) analyze portfolio duration under different stochastic processes, and FABOZZI, PITTS and DATTATREYA (1995) discuss various interpretations of portfolio durations. Similarly, convexity is described in BIERWAG, KAUFMAN and LATTA (1988) but not for bond portfolios.

This article analyzes the risk measures duration and convexity of a bond portfolio, and concludes that a market-weighted average of the specific durations and convexities calculated on the basis of the yield to maturity is superior to the calculation of duration and convexity from the portfolio yield to maturity based on the portfolio payments. Whereas the portfolio payments are ideal for calculating the portfolio yield to maturity, we show that using portfolio payments and the portfolio yield to maturity gives directly deceptive results regarding the risk measures duration and convexity when the term structure is non-flat.

Furthermore, the article shows that the slope of the term structure is essential to the size of the approximation errors following the calculation of portfolio duration and convexity based on the portfolio yield to maturity compared with zero coupon yields.

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2. Deriving Portfolio Duration and Convexity

Portfolio durations and convexities can be determined using two different angles. One way is to determine a market-weighted average of the individual durations and convexities related to the bonds in the portfolio, i.e.:

$$D_P = \sum_{i=1}^{N} \omega_i \cdot D_i$$

$$\omega_i = \frac{P_i}{\sum_{i=1}^{N} P_i} \qquad (1)$$

$$C_P = \sum_{i=1}^{N} \omega_i \cdot C_i$$

where D_p is the portfolio duration and C_p is the portfolio convexity. The weights ω are given by the market price (the present value) of each bond relative to the portfolio market price, and N indicates the total number of bonds in the portfolio. Finally, D_i and C_i are the individual durations and convexities for the i'th bond in the portfolio determined by:

$$D_i = \frac{\sum_{t=1}^{M} t \cdot c_{it} \cdot (1+z_t)^{-t}}{\sum_{t=1}^{M} c_{it} \cdot (1+z_t)^{-t}}$$

$$C_i = \frac{\sum_{t=1}^{M} t \cdot (1+t) \cdot c_{it} \cdot (1+z_t)^{-t}}{\sum_{t=1}^{M} c_{it} \cdot (1+z_t)^{-t}} \qquad (2)$$

where z_t is the zero coupon rate at time t representing a non-flat term structure, c_{it} is the cash-flow of bond i at time t, and M indicates the total number of cash-flows, which is equal to the time to maturity of the bond with the longest time to maturity.
In practice zero coupon rates are not always available and therefore as an alternative to (2) financial analysts and portfolio managers often use yield to maturity i.e.:

$$D_i = \frac{\sum_{t=1}^{M} t \cdot c_{it} \cdot (1+y_i)^{-t}}{\sum_{t=1}^{M} c_{it} \cdot (1+y_i)^{-t}}$$

$$C_i = \frac{\sum_{t=1}^{M} t \cdot (1+t) \cdot c_{it} \cdot (1+y_i)^{-t}}{\sum_{t=1}^{M} c_{it} \cdot (1+y_i)^{-t}} \qquad (3)$$

where y_i is the yield to maturity for bond i. This is usually how portfolio durations and convexities are calculated, although analysts are aware that this procedure only gives an approximation to the correct duration and convexity determined by combining (1) and (2).[1]
An alternative method to determine portfolio durations and convexities is to discount all portfolio cash-flows using the portfolio yield to maturity (the internal rate of return) as suggested by HAUGEN (1997), i.e.:

$$D_p = \frac{\sum_{i=1}^{N}\sum_{t=1}^{M} t \cdot c_{it} \cdot (1+y_p)^{-t}}{\sum_{i=1}^{N}\sum_{t=1}^{M} c_{it} \cdot (1+y_p)^{-t}}$$

$$C_p = \frac{\sum_{i=1}^{N}\sum_{t=1}^{M} t \cdot (1+t) \cdot c_{it} \cdot (1+y_p)^{-t}}{\sum_{i=1}^{N}\sum_{t=1}^{M} c_{it} \cdot (1+y_p)^{-t}} \qquad (4)$$

where y_p is the portfolio yield to maturity. It is easily shown that if the term structure is flat, $z_t = y_i = y_p$, we obtain the same portfolio duration and convexity in (1) as well as in (4) irrespective of using (2) or (3) in (1).
The following analysis compares the different methods in order to determine how close the approximations in (1)–(3) and (1)–(4) are to the correct duration and convexity in (1)–(2).

3. Assumptions

First, we assume the pricing on the bond market is consistent with the zero coupon term structure being flat in the initial situation at an interest rate of 7%. Secondly, we assume that the portfolio consists of 4 bonds with a coupon of 7% and maturities of 2, 5, 10 and 30 years, respectively. This means that all four bonds trade at a price of 1.000. Each bond represents 1/4 of the portfolio. Since the zero coupon term structure is assumed being flat and identical to the yield to maturity, the durations and convexities are identical for each of the bonds, irrespective of one using yield to maturity or zero coupon yields. In the same way the portfolio dura-

Table 1: Initial Portfolio

Coupon	Maturity	Weight	Price	Market Price	Yield to Maturity	Duration	Convexity
7	2 years	0.25	1.000	250	7.00	1.93	5.74
7	5 years	0.25	1.000	250	7.00	4.39	25.14
7	10 years	0.25	1.000	250	7.00	7.52	74.34
7	30 years	0.25	1.000	250	7.00	13.28	285.44
				1.000	7.00	6.78	97.66

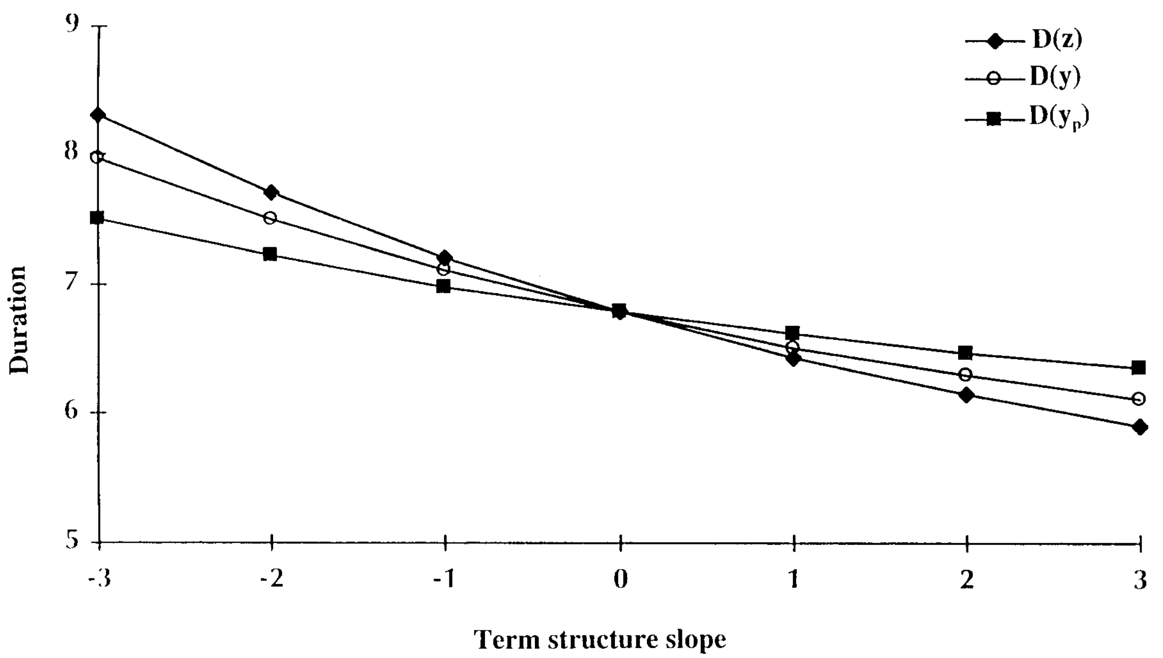
tion and convexity are equal, irrespective of one using a market price and duration weighted average (1) and (3) or the portfolio payments and the portfolio yield to maturity (4). In Table 1 the initial portfolio is illustrated.

From Table 1 we infer that in the case of a flat term structure at 7%, the portfolio duration is 6.78 and the portfolio convexity is 97.66.

4. Term Structure Simulations

It is widely held that a relatively good approximation will be achieved at the correct zero coupon calculated durations and convexities (2) by instead using the yield to maturity (3) in (1), also when the zero coupon curve is non-flat. This means that in practice duration and convexity calculations are

Figure 1. Durations at different slopes of the term structure



often based on the yield to maturity. When simulating different changes in the zero coupon term structure, it becomes possible to estimate the size of the approximation errors related to the use of yield to maturity in preference to zero coupon rates. Furthermore, such simulations can be used to estimate any approximation errors related to the use of the portfolio yield to maturity (4) in preference to a market price weighted average based on zero coupon rates obtained from (1) and (2). Based on a flat zero coupon term structure of 7%, the term structure will be changed in six different scenarios: partly increasing from 7% to 8%, 9% and 10%, respectively, partly decreasing from 7% to 6%, 5% and 4%, respectively, evenly distributed over the term structure from 0 to 30 years. Figure 1 illustrates the portfolio duration for the six scenarios calculated from zero coupon rates (1) and (2), which is indicated by D(z), the yield to maturity (1) and (3), which is indicated by D(y), and the portfolio yield to maturity based on portfolio payments (4), indicated by $D(y_p)$.

Thus, Figure 1 clearly shows a modest variation between the correct durations calculated from the zero coupon curve D(z), and the approximative durations based on the yield to maturity D(y). Contrary hereto, there is a significant difference to the durations calculated from the portfolio yield to maturity based on the portfolio payments $D(y_p)$. In Table 2 we present the portfolio in the case, where the term structure is assumed to be positively sloped with a slope of 3 percentage point, i.e. a rate of 7% at time 0 and 10% at the 30 years maturity.

From Table 2 we infer that the correct zero coupon based duration, D(z) and convexity, C(z) is 5.89 and 70.69, respectively, whereas the yield to maturity based duration, D(y) and convexity, C(y) is 6.09 and 77.71, respectively. Here we estimate the approximation error to be 3.4% concerning duration and 9.9% concerning convexity. Furthermore, Table 2 present the duration, $D(y_p)$ and convexity, $C(y_p)$ based on the portfolio payments and the portfolio yield to maturity as suggested by HAUGEN (1997). These are 6.33 and 85.04, respectively, amounting to an approximation error of 7.5% concerning duration and 20.3% concerning convexity.

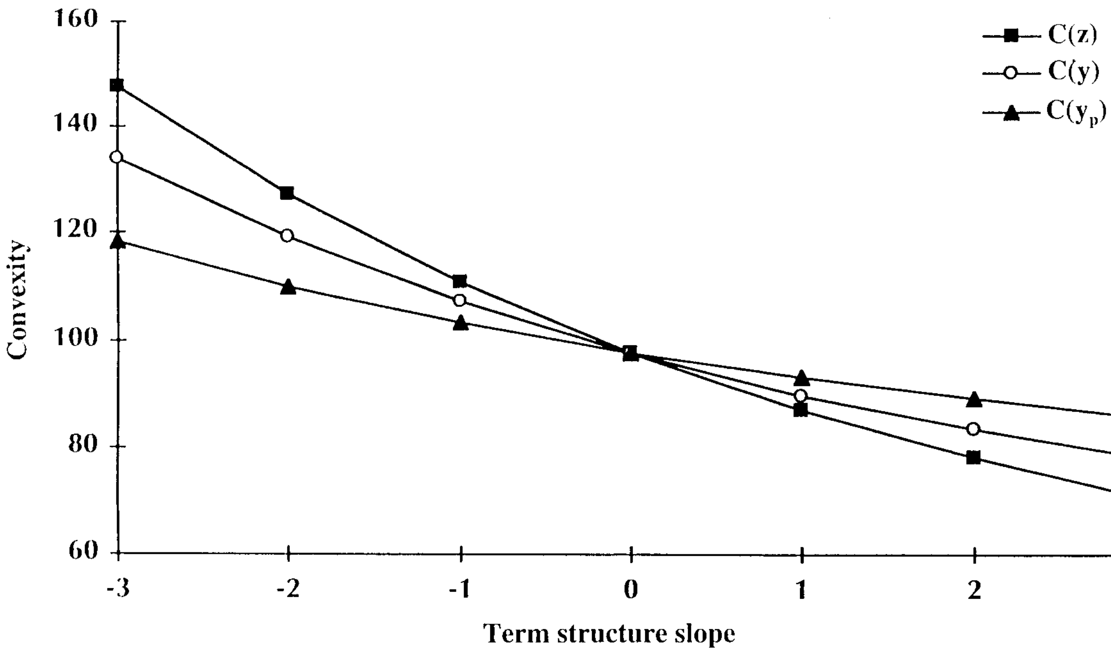
Furthermore, we infer from Figure 1 that the size of these approximation errors increases as the slope of the term structure either increases or decreases. Finally, Figure 1 demonstrates that if the term structure is positively sloped, the approximative durations overestimate the correct zero coupon based duration, whereas these approximations underestimate the true duration, if the term structure is negatively sloped (inverse).

An analogous conclusion can be drawn from Figure 2, showing the portfolio convexity at similar scenarios. Here we again conclude that the size of the

Table 2: Portfolio for a 3 percentage point positive term structure

Coupon	Maturity	Weight	Value	Market Value	Yield to Maturity	Duration D(z)	Convexity C(z)
7	2 years	0.25	996.5	249.13	7.19	1.93	5.74
7	5 years	0.25	980.9	245.23	7.47	4.38	25.07
7	10 years	0.25	940.4	235.10	7.88	7.40	72.71
7	30 years	0.25	808.4	202.10	8.84	10.86	203.74
				931.56	8.17	5.89	70.69
Duration and convexity, D(y) and C(y)						6.09	77.71
Duration and convexity, $D(y_p)$ and $C(y_p)$						6.33	85.04

Figure 2. Convexities at different slopes of the term structure



approximation errors is increasing as the slope of the term structure either increases or decreases, as well as the true portfolio convexity is overestimated for a positively sloped term structure and vise versa.

The reason for these approximation errors is the use of yield to maturity, which assumes a flat term structure. When the term structure is non-flat, the yield to maturity will not be suitable for calculating durations and convexities for portfolios.

At this point an interesting question is as to why the deviation between D(z) and D(y) is significantly smaller than the deviation between D(z) and D(y_p)? However, the fact is that using the yield to maturity of the portfolio y_p, one assumes a flat term structure for the overall portfolio, whereas using the yield to maturity of each bond y_i, when deriving the individual durations and convexities, one only assumes a flat term structure within each bond but not for the overall portfolio. In this latter case a non-flat zero coupon term structure to some extent will be reflected in the calculation of the market-weighted average durations and convexities, since when the term structure is positively sloped, the yield to maturity of the individual bonds will increase with the time to maturity. In the case with an increasing term structure of 3 percentage points, the zero coupon interest rate is 7.20% in the second year, 7.50% in the fifth year, 8.00% in the tenth year and 10% in the 30th year, while the yield to maturity is 7.19% for the 2 year bond, 7.47% for the 5 year bond, 7.88% for the 10 year bond and 8.84% for the 30 year bond, respectively. The increasing yield to maturity for the four bonds thus to a certain extent reflects the positively sloped zero coupon term structure contrary to the portfolio yield to maturity, which is 8.17%, implying a flat yield curve. If in this case the duration is calculated using the portfolio yield to maturity of 8.17% based on the portfolio payments, the duration of the portfolio will thus be overestimated by (6.33 – 5.89) 0.44. A miscalculation of this size may have a drastic influence on portfolio management.

5. Summary

The general opinion is that as a rule of thumb yield to maturity may be used instead of zero coupon interest rates, when calculating risk measures such as duration and convexity for non-callable bond portfolios. This analysis has shown that the rule of thumb gives rise to considerable approximation errors when the term structure is non-flat.

Furthermore, the analysis has challenged the argument by HAUGEN (1997), who argues that portfolio durations and convexities can be obtained by using portfolio payments and the portfolio yield to maturity. We have shown that HAUGENS'S method gives rise to approximation errors significantly higher than the approximation errors that one obtain if portfolio durations and convexities are derived using a market-weighted average of the individual durations and convexities based on the yield to maturity of each individual bond in the portfolio.

This result is due to the fact that the portfolio yield to maturity assumes a flat term structure for the overall portfolio, whereas using the yield to maturity of each bond one only assumes a flat term structure within each bond. A non-flat zero coupon term structure will therefore be reflected in the calculation of the market-weighted average durations and convexities, since it is reflected in the yield to maturity of the individual bonds.

Appendix

Portfolio duration and convexity for a non-flat term structure

If c_{it} and c_{jt} is the cash-flows of two bonds i and j at time t, and z_t is the zero-coupon rate at time t, the duration of the two bonds can be determined as:

$$D_i = \frac{\sum_{t=1}^{M} t \cdot c_{it} \cdot (1+z_t)^{-t}}{\sum_{t=1}^{M} c_{it} \cdot (1+z_t)^{-t}}$$

$$D_j = \frac{\sum_{t=1}^{G} t \cdot c_{jt} \cdot (1+z_t)^{-t}}{\sum_{t=1}^{G} c_{jt} \cdot (1+z_t)^{-t}} \quad \text{(A.1)}$$

and the duration of a portfolio consisting of the two bonds with equal weights is given by (M > G):

$$D_{i+j} = \frac{\sum_{t=1}^{M} t \cdot (c_{it} + c_{jt}) \cdot (1+z_t)^{-t}}{\sum_{t=1}^{M} (c_{it} + c_{jt}) \cdot (1+z_t)^{-t}} \quad \text{(A.2)}$$

Now defining the market price of each bond to be given by the standard present value relationship, we have:

$$P_i = \sum_{t=1}^{M} c_{it} \cdot (1+z_t)^{-t}$$

$$P_j = \sum_{t=1}^{M} c_{jt} \cdot (1+z_t)^{-t} \quad \text{(A.3)}$$

and the portfolio duration in (A.2) can be rewritten using (A.3) to be:

$$D_{i+j} = \frac{\sum_{t=1}^{M} t \cdot c_{it} \cdot (1+z_t)^{-t}}{P_i + P_j} + \frac{\sum_{t=1}^{M} t \cdot c_{jt} \cdot (1+z_t)^{-t}}{P_i + P_j} \tag{A4}$$

$$= \frac{P_i}{P_i + P_j} \cdot D_i + \frac{P_j}{P_i + P_j} \cdot D_j$$

Similarly, one can along the same lines prove that this relationship holds for the portfolio convexity as well, i.e.:

$$C_{i+j} = \frac{P_i}{P_i + P_j} \cdot C_i + \frac{P_j}{P_i + P_j} \cdot C_j \tag{A.5}$$

Therefore, the portfolio duration and convexity can be determined as a simple weighted average of the individual durations and convexities, respectively, with weights equal to the market price of each bond relative to the total portfolio market price, which is equivalent to (1) in the case, with only two bonds in the portfolio.
This relationship holds for any shape of the term structure as long as the individual durations and convexities are based on zero coupon rates. If instead yields to maturity are used calculating durations and convexities, (A.4) will only be valid if the term structure is flat i.e. the yield to maturity being equal for all bonds.

Footnote

[1] This is proved in the appendix for a non-flat term structure. The proof is a generalization of the proof in BIERWAG (1987), who assumes a flat term structure.

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