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Credit Risk Pricing: A Literature Survey

1. Introduction

Credit risk has always been a major topic of concern for banks and other financial intermediaries. Up until recently, management of credit risk was mostly done by credit risk departments that helped assign credit limits to the different counterparties. Such credit limits take poorly into account the evolution of the risk considered or the real impact of instruments that have become more and more complex. Derivatives for example are widely used. But more than 50% of derivatives used are OTC products and thus present some credit risk (OTC derivatives are growing at a much faster pace than exchange-traded derivatives too).
As their risk management systems get more sophisticated with respect to market risk, financial institutions become more aware of the weakness of their credit risk exposure calculation and of the need to value this exposure, rather than ration it through credit lines.

*This is a survey of literature and does not represent original work. This article would not have been written without the work of the many academics quoted in reference. Discussions with many have also been useful (notably with Hayne Leland, Suresh Sundaresan, Hugues Pirotte, and many professionals in banks). I also thank Manuel Ammann (the referee) for his help. I remain solely responsible for possible mistakes. Professor Didier Cossin, HEC, University of Lausanne, tel: ++41 - 21 - 692 34 69, fax: ++41 - 21 - 692 33 05, email: Didier.Cossin@hec.unil.

At the same time, the most sophisticated players in the field have started trading credit risk derivatives that by themselves allow for a better management of credit risk exposures. These credit risk derivatives (such as credit swaps, default swaps or puts, total return swaps, etc.) also require some form of pricing. An easy arbitrage-free pricing on the basis of a simple instrument such as a bond can not always be done. Competitive pressure gives thus the most advanced players a nice comparative advantage.
Hence both the need for better credit risk management and for a better understanding of new instruments coming to the markets require the help of good theories of credit risk pricing.
Many theoretical developments have indeed appeared in this field during the last few years. The goal of this paper is to present the origins and current trends in research on credit risk pricing.
The paper proceeds first by reviewing briefly actuarial methods used in credit risk pricing, then presents the contingent claim paradigm used widely in the field, before stressing problems with real world applications. The two last sections analyze the latest evolution of the models, notably the combination of interest rate risk and credit risk in models with stochastic interest rates, as well as the alternative models with exogenous bankruptcy process that have recently appeared.

2. Actuarial Methods for Valuing Credit Risk

One branch of research, mostly developed in banks'research departments, bases itself on actuarial calculations and probabilistic mathematics to infer a pricing of default from historical data. (See for example IBEN and LITTERMAN (1991), ALTMAN and KAO (1992), LUCAS and LONSKI (1992), IBEN and BROTHERTON-RATCLIFFE (1994), and SORENSEN & BOLLIER (1994). See also for a critical approach DUFFEE (1995a and 1995b)).
Although these methods are widely used in banks, they present major difficulties that doom them in many dimensions. This part of the review will thus be brief.
The basic principle of this type of approach is to estimate the probability of default (or of rating downgrade) and to estimate (often independently) the value of the contract at possible default times.
Rating agencies are standard sources for default probabilities. Techniques used to forecast default probabilities for individual firms are described in ALTMAN, HALDEMAN and NARAYANAN (1977). Methodologies have evolved from the calculation of mortality rates to the calculation of rating category migration probabilities. These probabilities (usually organized in so-called transition matrices) consist in the probabilities of downgrade and upgrade by rating category. These calculations are now frequently used by professionals.
As stressed by DUFFEE (1995a), end users tend to develop MonteCarlo simulations without taking into account the uncertainties in the models used to generate the estimates. Second they rarely take into account the correlation among probabilities of default and estimates of possible losses. These correlations certainly affect the results. One can expect for example exposures linked to derivatives to rise with the volatility of the markets. But it is also at such a time that probabilities of default will rise. Unfortunately, historical correlations are difficult to obtain empirically. Some try to overcome this difficulty by using advanced analysis methods such as neural networks (see for example TRIPPI and TURBAN (1996)).
Third these types of models often fail to consider the impact of the total portfolio of the institution considered on the upper bounds of credit losses associated with a single instrument or portfolio of instruments. By neglecting correlation effects, they thus obtain results that are not only based on simple replication of historic conditions but also that do not support the experience of the institution itself.
Much of the recent advances in this area are trying to address the porfolio issue, either at a practical and descriptive level of the portfolio of credit risks in a firm (for example with the help of CreditMetrics™) or at a more theoretical level to try and find a MARKOWITZ type of efficient frontier with credit risks. Credit risky returns have the particularity of not presenting the statistical properties necessary to apply the MARKOWITZ framework (lognormality) and alternative models of an efficient frontier have thus to be found (see for example ALTMAN and SAUNDERS (1997)).
Nonetheless, as explained and illustrated underneath, all these methods face the major difficulty of being strongly dependent on historical estimates of credit risk dynamics.

3. The Contingent Claim Paradigm Applied to Credit Risk

A major difficulty of actuarial methods is their complete dependence on historical data. They consist in fitting expectations of default to default data of the past. Their results are thus not coherent with the evolution of fundamentals across time. It is well known for example that fitting by taking an average of call values on past data, even calculating call values on expectations of stock values based on historical averages of stock prices, will not give rational call prices. Instead an arbitrage-free theory of option pricing has been developed that relates call prices to current market variables. One can from there on differentiate between the model(s) proposed and the estimates of the variables calculated to look for mistakes and approximation in call prices. Similarly, a rational theory of credit risk based on financial economics was developed as early as 1974 as an application of contingent claim analysis.

3.1 The Original Contingent Claim Analysis (CCA) Framework: MERTON (1974)

Contingent claims analysis (option pricing) can be used to value the component parts of a firm's liability mix. In general, the value of each component will depend upon the stochastic variables which determine the evolution of the firm's asset value, the evolution of the interest rate, the payouts (dividends, coupons, etc.) to the various claimants, and the division of the firm at any point of reorganization (e.g., bankruptcy). MERTON (1974) starts with a simplified model that yields useful insights and shows the way to more complete (and more complex) valuation.

In short, the idea is to use option pricing to value the default risk spreads of fixed income instruments. Hence the description of a „risk structure of interest rates" that completes the traditional „term structure of interest rates". The method makes it possible to analyze and measure the impact on credit risk spreads of a change in asset volatility, a change in interest rates volatility, different maturities of debt, etc.

Let $V(t) \equiv$ market value of the firm at time t.

V(t) thus represents the sum of the value of the different liabilities of the firm, such as straight debt, convertible debt, common stocks, etc.

The usual assumptions made in CCA literature are supposed to hold, that is:

A.1 Perfect markets: The capital markets are perfect with no transaction costs, no taxes, and equal access to information for all investors.

A.2 Continuous trading.

A.3 Short sales of all assets are allowed.

A.4 ITO dynamics: V(t) follows the dynamics:

$$\frac{dV}{V} = \alpha(V,t)dt + \sigma(V,t)dz \qquad (1)$$

where $\alpha(V,t)$ is the instantaneous expected rate of return, $\sigma^2(V,t)$ the variance of the return on the underlying assets and dz a standard Wiener process. (Note the special case where $\alpha(V,t) = \alpha V - C$ where C is total cash outflow per unit time).

A.5 Shareholder wealth maximization: Management acts to maximize shareholder wealth.

Two other assumptions are technical and can be easily relaxed (from classical option pricing theory):

A.6 Constant σ^2

A.7 Nonstochastic term structure: the instantaneous rate r(t) is a known function of time.

Suppose that the firm has two classes of securities: A single homogenous class of zero-coupon discount bonds, with face value B and maturity T, and equity. The indenture („terms") of the bond issue contains the following simplified event of default covenant: In the event that the face value payment is not met, the bondholders receive the entire value of the firm and the owners of the firm receive nothing. In this framework, the firm is prohibited from issuing any new senior claims on the firm nor can it pay dividends or repurchase shares prior to the maturity of the debt (extensions can be dealt with).

Hence the value of the bond at maturity is

Min (V(T), B)

Denote	D(V, t; T, B)	the value of the bond issue at time t for V(t) = V
and	E(V, t; T, B)	the value of the equity at time t for V(t) = V.

(Short notations: D(V,t), E(V,t))

We have that:

$$V \equiv D(V, t; T, B) + E(V, t; T, B) \tag{2}$$

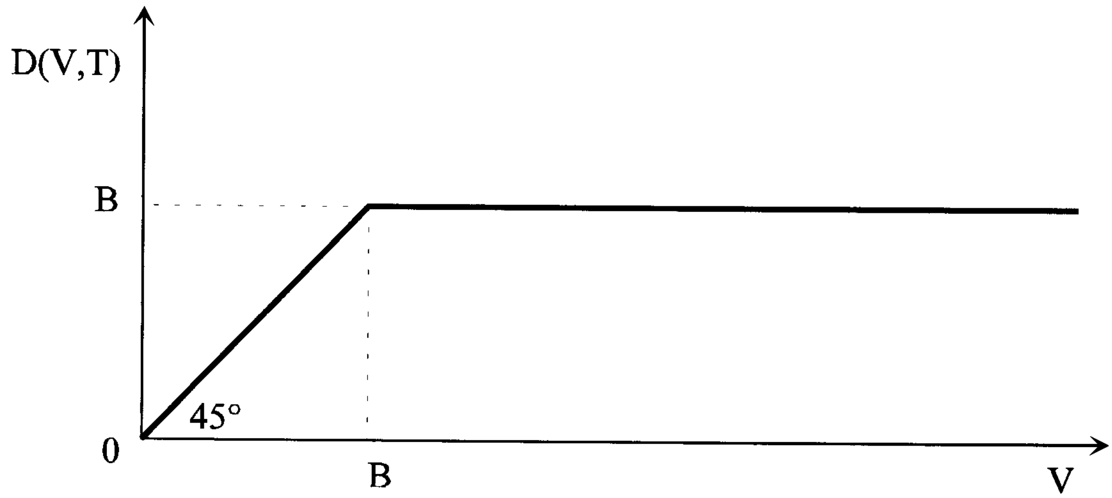
We have the payoffs'table at time T:

Firm Value	Bond Value	Equity Value
$V(T) \leq B$	$V(T)$	0
$V(T) > B$	B	$V(T) - B$

$$\begin{aligned} D(V,T) &= \min(V(T),B) \\ &= V(T) - \max(0, V(T) - B) \\ &= B - \max(0, B - V(T)) \end{aligned} \tag{3}$$

From (3), the terminal payoff to debt is functionally equivalent to owning the assets of the firm and being short a call option on those assets with an exercise price of B. (Alternatively, the debtholders can be considered to have lent money risklessly with face value B and gone short a put option on the assets of the firm with an exercise price of B).

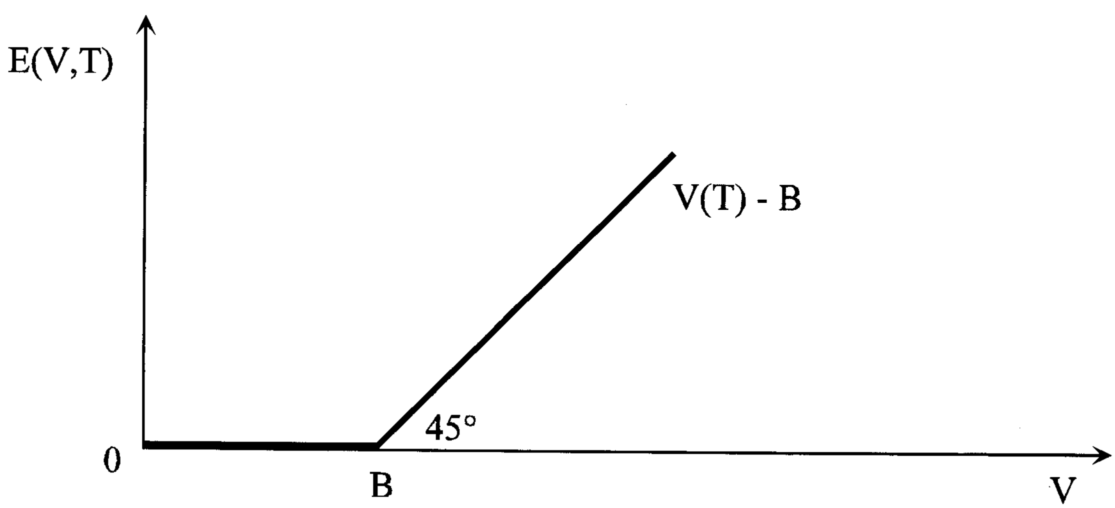
Figure 1



Similarly we have for the value of the equity:

$$E(V,T) = \max(0, V(T) - B) \tag{4}$$

Figure 2



The payoff structure to the levered corporate equity is isomorphic to the one for a call option on a share of stock where the maturity date of the firm's debt T corresponds to the expiration date of the option, the promised payment on the debt B corresponds to the exercise price of the option and the firm value V corresponds to the underlying security. We can thus use option pricing to value credit risky bonds.
As usually in CCA, the value of the equity is given by the PDE:

$$0=\frac{1}{2}\sigma^2(V,t)V^2E_{vv}(V,t)+rVE_V(V,t) -rE(V,t)+E_t(V,t) \qquad (5)$$

s.t.

$E(V,t)/V \leq 1$ (5a)
$E(0,t) = 0$ (5b)
$E(V,T) = \max(0, V - B)$ (5c)

In the special case σ^2 = constant, we have the classical BLACK and SCHOLES results:

$$E(V,t;T,B)=VN(h_1)-Be^{-r(T-t)}N(h_2) \qquad (6)$$

with $h_1=\dfrac{\log\frac{V}{B}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$

$h_2=h_1-\sigma\sqrt{T-t}$

$N(y)=\dfrac{1}{\sqrt{2\Pi}}\displaystyle\int_{-\infty}^{y}e^{\frac{-u^2}{2}}du$

Standard Normal Cumulative Density Function

The value of the debt will also satisfy the same PDE, but with the corresponding boundary conditions:

$$0=\frac{1}{2}\sigma^2(V,t)V^2D_{vv}(V,t)+rVD_V(V,t) -rD(V,t)+D_t(V,t) \qquad (7)$$

s.t.

$D(V,t)/B \leq 1$ (7a)
$D(0,t) = 0$ (7b)
$D(V,T) = \min(V(T), B)$ (7c)

In the special case where σ^2 = constant, we know that the value of a risky 0-coupon bond is equal to the value of the firm less the value of the equity (the call option) or:

$$\begin{aligned}D(V,t)&=V-VN(h_1)+Be^{-r(T-t)}N(h_2)\\&=VN(-h_1)+Be^{-r(T-t)}N(h_2)\end{aligned} \qquad (8)$$

as $N(-y) = 1 - N(y)$

Hence:

$D(V,t)=VN(k_1)+Be^{-r(T-t)}N(k_2)$
where $k_1=-h_1, k_2=h_2$

Note that under the risk neutral process, $N(k_2)$ is the probability that the firm will be solvent when the bond matures. Thus, the second term in (8) is the riskless discounted expected value of receiving the promised payment B in full. The first term is the present value of receiving all the assets of the firm conditional on their being worth less than B. Note that the comparative statics for a 0-coupon bond can be determined from those of options.

3.2 The Risk Structure of Interest Rates

It is common in dealing with bonds to discuss them in terms of yields rather than prices. The yield-to-maturity of a discount bond in a continuous time framework is the solution to:

$$D(V,t)=Be^{-R(T-t)} \qquad (9)$$

[We have $R \geq r$ as $D(V,t)\leq Be^{-r(T-t)}$]

The yield spreads or risk premium is the difference between the yield and the riskless rate. We have in the case σ = constant:

$$R(d,t;T,r) = r - \frac{1}{T-t}\log\left(N(k_2) + \frac{1}{d}N(k_1)\right) \quad (10)$$

$$s(d,t;T,\sigma) = -\frac{1}{T-t}\left(\log N(k_2) + \frac{1}{d}N(k_1)\right) \quad (11)$$

where $d = \frac{Be^{-r(T-t)}}{V}$

d is the quasi debt-to-firm value ratio (debt to firm value ratio if debt was riskless).

(11) shows that the frequently used credit risk premium of a bond of a given maturity is a function of two and only two major variables (under the assumption of a known term structure): the volatility of the firm overall value and a form of leverage ratio that is the promised payment ratio to the value of the firm. It can be shown that the risk premium is an increasing function of the quasi debt ratio, as one would intuitively expect, and of the volatility of the firm. As usual in option pricing, but seemingly paradoxal for first users, the rate of return on the underlying security (here the growth rate in the value of the firm) has no impact on the credit spread.
Another natural way to measure the bond's risk is by its instantaneous standard deviation of returns. It can be shown that this standard deviation is (using ITO's lemma):

$$\begin{aligned}\sigma_D(d,t;T,\sigma) &= \frac{VD_V}{D}\sigma \\ &= \eta\left(d,\sigma^2(T-t)\right)\sigma\end{aligned} \quad (12)$$

where

$$\eta\left(d,\sigma^2(T-t)\right) = \frac{V\frac{\partial D}{\partial V}}{D} = \frac{N(k_1)}{N(k_1) + dN(k_2)}$$

η is the elasticity of the bond price relative to that of the firm as a whole.

The standard deviation of the return on the bond as defined above represents the risk of the rate of return over the next trading interval. It is thus a different measure of risk from the spread as defined above. The bond's standard deviation measures the risk over the next instant. The yield spread, on the other hand, is the promised risk premium over the remaining life of the bond. Nonetheless, the standard deviation of the return on the bond depends on the same variables as the spread, notably maturity, quasi debt ratio and volatility of the firm. It is interesting to understand which of these two measures of risk is more valid in which environment, specially as practitionners tend to use spreads to compare bonds' riskiness.

3.3 Comparative Statics

These analytical formulas (or the numerical analysis of more complex models) give us the ability to study the impact of changes in debt ratio, changes in volatility of the firm and changes in the maturity of the debt on the credit spreads or on the instantaneous standard deviation of the debt returns, which are two measures of risk of the debt that are not equivalent (see MERTON (1974)). One of the most interesting theoretical results of MERTON (1974) consists in the impact of a longer maturity on the two measures of risk. The effect of a longer maturity is indeed not clear: the yield spread can either rise or fall. The spread decreases in maturity if $d \geq 1$ (with d the quasi-debt ratio). If $d < 1$, the spread first rises and then falls while the risk is rising.
If $d > 1$, the firm is technically insolvent. To avoid bankruptcy, it will need to have pleasant earnings surprises. As $T - t \to \infty$, the instantaneous risk approaches the limit $\sigma/2$ and the yield spread vanishes, as there is more time for the pleasant surprises to happen.
If $d < 1$ and the bond has only a short time to go before maturity, it is unlikely there will be a default of the bond. As maturity increases, the likelihood of default increases and the yield spread

widens. For continued increases in maturity, the instantaneous risk continues to rise to its limit but yield spread begin to fall (as there can never be a default on a perpetual bond: $\lim_{T-t\to\infty} s = 0$).

Hence the special case of high leverage firms. Longer maturity need not make debt riskier.

This result shows that *the yield spread does not necessarily reflect accurately the relative default risks of two bonds of different maturities.* The two measures of risk will not agree on which bonds are riskier.

3.4 Some Empirical Investigations of the MERTON Model

JONES, MASON and ROSENFELD (1984) attempted a true test of the MERTON technology. Such a test faces large difficulties: Most firms' capital structures consist of many classes of equity and fixed income products (preferreds, callable convertibles, callable non convertibles, bonds with sinking fund requirements, etc...). The multiplicity of the issues itself can create major difficulties when looking at the interaction not only of default rules but of call policies that will both affect the pricing. The authors thus use a sample of companies with relatively simple capital structures. They make simplifying assumptions (and notably use a deterministic term structure). They find results that show the superiority of the CCA model over a naive (riskless) model for non-investment grade bonds but not for investment grade bonds. The authors find low theoretical spreads compared to actual spreads. They do not compare the model to its true competitors, the actuarial models. They also lack some of the modern refinements that would make the CCA model more realistic (stochastic interest rates notably). They nonetheless show how to fully implement the CCA methodology in order to apply it to credit risk measurement.

Although not a true test of the MERTON model, the SARIG and WARGA (1989) paper presents empirical results that seem to confirm the comparative statics obtained by MERTON. The authors analyze 137 corporate issues of zero coupon bonds representing 42 different companies. They measure the yields' spread of corporate bonds above the yield on Treasury strips of the same maturity, a traditional credit risk measure, and, as seen above, one for which MERTON derived theoretical comparative statics. 15 years after MERTON's paper, the authors find an empirical shape that corresponds very clearly to the theoretical predictions of the model.

Interestingly, the behavior of comparative statics remain very similar to those in the simple MERTON model when the models are made more complex and integrate stochastic interest rate structures (see SHIMKO and alii (1993) and LONGSTAFF and SCHWARTZ (1995)). The basic intuition of the MERTON model seems thus to be useful for pricing risky debt.

4. Problems with Real World Applications

MERTON's paper presents an extreme simplification of the real word: a firm with a unique 0-coupon debt issue. INGERSOLL (1987), notably, shows that the model can be extended to many other cases (sections 4.1 to 4.4 borrow heavily from this work). These extensions are mostly engineering of more realistic financial structures than a single 0-coupon issue. Deeper difficulties (stochastic interest rates, different bankruptcy rules) are approached later.

4.1 Coupon Bonds

Risky coupon bonds *cannot* be priced as if they were a portfolio of risky pure discount bonds. Take the same framework as before with the modification that coupon payments occur continuously at a rate per unit of time[1] C. The coupon bond will satisfy the PDE:

$$0 = \frac{1}{2}\sigma^2 V^2 D_{VV} + (rV - c)D_V - rD - D_t + C$$

subject to the same boundary conditions as before. The well known closed formula obtained in the perpetual case gives the pricing of preferred stocks with no maturity date. Numerical analysis will be necessary for other cases.

4.2 Subordinated Debt

Firms rarely having a single issue of debt, priority rights to the assets in case of default affect the pricing the debt. For example, suppose a firm has 2 outstanding shares 0-coupon bonds maturing at the same time T.

B dollars are promised on the senior debt

b dollars are promised on the junior or subordinated debt.

The senior debt has absolute priority to all the assets of the firm. The value of the junior bond and the senior bond together is equal to that of a single bond with a face value of B + b. So that:

$$J(V,t;T,B,b) = D(V,t;T,B+b) - D(V,t;T,B)$$

The comparative statics of the junior debt can be determined from these. It could be shown for example that when the value of the firm is low, the junior debt tends to behave like equity while when the value of the firm is high, it has more pronounced debt characteristics.

Different Maturities and Cross Default Condition:

When the junior debt matures first and no provisions have been made, the junior debt can effectively be senior. One common way to maintain some priority to senior debt is a cross default indenture. Such a protection is only partial and specific valuation needs to be made (see INGERSOLL (1987) p. 426–429). See also the treatment of secured junior debt as approached in INGERSOLL (1987) p. 429. The analysis of the different types of debt can be made in a traditional way, starting from the payoff functions at maturity of the different debt components.

4.3 Convertible Securities

A common type of securities issued by corporations is a convertible bond. A convertible bond has one or more fixed payments like a regular bond but can also be converted in a certain number of shares of common stocks of the company.
Suppose the firm has a capital structure with a single convertible bond issue and common equity. If the bondholders choose not to convert, they will receive min (V(T), B). If they convert, they will receive n new shares of common stock that are worth $\frac{n}{N+n}V(T)$, where N is the current number of shares outstanding. Define $\gamma \equiv \frac{n}{n+N}$.

We have the payoff at maturity:

$$C(V,T;T,B,b) = \begin{cases} \gamma V(T) & \text{if } B \le \gamma V(T) \\ B & \text{if } B \le V(T) < B/\gamma \\ V(T) & \text{if } V(T) < B \end{cases}$$

Hence $C(V,T) = \text{Min}(V(T), B) + \text{Max}(\gamma V(T) - B, 0)$ which is the sum of a bond payoff and a warrant payoff (or a call on γV).
In the case of a non dividend-paying stock, it can be shown that no conversion happens before maturity and, in the case σ = constant:

$$C(V, t) = D(V, t) + BS(\gamma V, t)$$

where $BS(\gamma V, t; T, B)$ is the BLACK and SCHOLES value of a call on γV with maturity T and exercise price B.
Other convertible securities such as convertible preferred stocks can be analyzed in a similar man-

ner. Similarly, closed form solutions are often easily attainable for non-dividend paying stocks, as the conversion policy is simpler (do not exercise before maturity, just as with any other call option). In the case of dividend paying stocks, the results will be obtained through numerical analysis in most cases and will depend on the cash payments of the stocks (the dividends) as well as on the cash payments from the convertible securities (the coupons in the case of a bond).

4.4 Callable Bonds

Most bonds that are issued are callable. A typical coupon bond issued by a corporation may not be callable for the first 5 or 10 years (call protection period), and then be callable at a price which declines overtime until it reaches the face value at maturity. The call option on a bond is just the same as any other call option. A call provision on a convertible bond is slightly different because it may force conversion, hence shortening the maturity of the conversion option that convertible bondholders have. For a zero coupon convertible with a constant call price equal or greater than its face value, the proper call policy for the firm is to force conversion as soon as possible. On the other hand, the bond should never be called when bondholders will take the cash payment.

Contingent claim analysis will allow to establish the optimal call policy and then to value the instrument, by solving the usual partial differential equation with boundary and terminal conditions well defined. When closed form solutions are not known, numerical solutions can be easily obtained.

The empirical literature has recently analyzed whether callable bonds are actually called optimally by firms. Of course, the actual call policy should affect the pricing rather than the theoretical one. Recent results tend to suggest that actual and theoretical call policies may actually be closer than what had previously been believed (see ASQUITH (1995)).

4.5 Swaps

COOPER and MELLO (1991) focus on pricing credit risk in swaps. The added complication to modelling credit risk in swaps compared to classical models of credit risk is that the defaulting counterparty may not be due to make any payment (even if it was non-defaulting) given the evolution of the underlying market. In particular, if exchange or interest rates move in a way that the net value of the swap is positive for the defaulting counterparty, there may not be any cost at being defaulted on.

Throughout their paper, COOPER and MELLO (1991) make the following assumptions: Swaps are subordinate to debt in bankruptcy; In the event of a default on its debt by a counterparty that is owed value in a swap, the value of the swap will be paid to the bankrupt firm; There is only one risky counterparty. COOPER & MELLO derive the relationship between swap market default spreads and debt market default spreads where default spreads for the fixed rate and variable rate debt markets are defined analogously to MERTON (1974). The authors then analyze three possible treatments in default and their wealth-transfer impacts (a work extended by BAZ (1995)). The authors make assumptions on the stochastic processes followed by the value of the firm, and the variable swap payment. They thus obtain equilibrium swap spreads.

The major flaw of the model for actual swap credit risk pricing is that it remains a one-sided default risk model. Stochastic interest rates should also be introduced, as shown later.

4.6 Computing the necessary inputs

1. Estimating the Value of the Firm

If all claims are publicly traded, then the value of the firm can be observed and prices for all claims, relative to the observed firm value, can be predicted. When all claims are not publicly traded, an alternative approach has to be taken. For example, the total value of all traded claims can be used to infer firm value. The analysis brings out the firm value that is consistent with the observed value of all traded claims. This implied firm value is then used to predict bond prices (iteration may well be necessary). (See RONN and VERMA (1986) for a methodology that extracts firm value dynamics from traded equity dynamics.)

2. Estimating the Standard Deviation for Each Firm

Two procedures can be used to estimate the standard deviation for each firm.

The first procedure is based on forming a monthly time series for the value of the firm using (e.g.) 24 trailing months of data. The value of the firm is estimated as the sum of the market value of equity, the market value of traded debt and the estimated market value of nontraded debt. The market value of the nontraded debt is estimated by assuming that the ratio of book to market was the same for traded and nontraded debt. The logarithmic total return on the value of the firm, including any cash payouts/payins is calculated and the standard deviation of these returns determined.

The second procedure is a maximum likelihood procedure based on the relationship between the standard deviation of the return to the firm and the equity. Given the assumptions of CCA, it follows from ITO's Lemma that the instantaneous standard deviation of equity, σ_E, is given by:

$$\sigma_E = \sigma_V E_V V / E$$

where σ_V is the standard deviation of the return to the firm and E_V is the partial derivative of the value of equity with respect to the value of the firm. The method II procedure is to run the model using the method I estimate of standard deviation as a seed. The value of the firm, V, the value of the equity, E, and the partial derivative of equity, E_V, with respect to the value of the firm which are implied by the observed total value of marketable claims are read from this first pass of the model. Then the standard deviation of return to the equity is calculated, using market data, over a period immediately preceding the test date. Given (3), a new estimate of σ_V using σ_E, E, V and E_V. The model is then rerun using the new estimates of σ_V.

In general, CCA can be used to price credit risk on any instrument, thanks to the intense engineering that has been going on in the 1980s. Obviously though, analysis can become very complex when the capital structure of the firm becomes complex. Also, two problems that were major impediments to real life applications have been solved only recently: stochastic interest rates and complex bankruptcy rules.

5. Credit Risk, Stochastic Interest Rates, and Other Bankruptcy Rules

The simple original model that was used for developments on specific instruments has two major flows, that can be dealt with more or less successfully. The first one is the assumption of constant interest rates, an assumption that is quite troublesome: One would expect the dynamics of interest rates to actually have an impact on credit risk. The second one is the oversimplified bankruptcy rule assumed in the original model. Violation of priority rules are for example common place in bankruptcy, and securities rarely obtain after bankruptcy procedures the exact amount they should get. These two problems are addressed in the following sections.

5.1 Stochastic Interest Rates

An obvious major flaw of the original MERTON (1974) is the assumption of constant interest rates. Basis risk has to be combined to credit risk in order to explore critical issues of pricing and /or management such as:

- How can one value a fixed income instrument in the presence of both credit risk and basis risk?
- How does the correlation between a bank's credit risk and interest rate movements affect its borrowing cost?
- What maturity debt (or face value) should a corporate treasurer issue to minimize fluctuations in the value of the corporation's stock price?
- How much capital should be allocated to activities within a bank that vary both in absolute degree of credit risk and in the correlation of that risk with movements in interest rate risks?

Consider the basic framework of section 3 again. Assume also that the risk-free term structure is consistent with the VASICEK (1977) model. The VASICEK model assumes that the short-term riskless interest rate is mean-reverting to long run mean γ at speed k and that its instantaneous volatility σ_r is constant:

$$dr = k(\gamma - r)dt + \sigma_r dz_r$$

The VASICEK model unfortunately allows for the possibility of negative interest rates to arise. On the other hand, HULL and WHITE (1992) have shown that the modified VASICEK model can be used to fit any observable term structure (while the COX-INGERSOLL-ROSS (1985) model cannot).

The price of a zero-coupon riskless bond can be derived as in VASICEK (1977). Assume also $dzdz_r = \rho dt$

The value of a risky debt is a function of 2 stochastic factors V and r. Using ITO's lemma and the standard no arbitrage argument, SHIMKO and alii (1993) show that D must satisfy the PDE:

$$0 = D_t + \frac{1}{2}D_{vv}V^2\sigma^2 + \frac{1}{2}D_{rr}\sigma_r^2 + D_r V\rho\sigma_r\sigma V$$
$$+D_r\left(k(\gamma - r) - \lambda\right) - rD + rD_V V$$

The value of risky debt when interest rates are stochastic can be written:

$$D = V - VN(l_1) + BP(T-t)N(l_2)$$

with P the value of the riskfree debt and other variables as defined in SHIMKO and alii (1993).

(For derivation of the PDE and its resolution in the case of the COX-INGERSOLL-ROSS model of term structure, see for example TITMAN & TOROUS (1989)).

Just as we did in the simple discount bond, constant interest rate model, we can derive here the comparative statics. It is interesting to note that the classical MERTON result of the bell shape relationship of spreads to maturity still obtains.

5.2 Other Bankruptcy Rules

LONGSTAFF and SCHWARZ (1995) develop a model similar to SHIMKO and alii (1993) except for the bankruptcy procedure. While in most papers using option pricing frameworks bankruptcy time is defined as the moment when *V*, the value of the firm, reaches *D*, the value of the debt, the authors use a threshold value *K* defined similarly to BLACK and COX (1976). Default occurs when V reaches K which gives more flexibility to define the time of financial distress in the calculation. X, the ratio of V over K, is a sufficient statistic for the riskiness of the firm. Also, the authors do not assume perfect application of priority rules under financial distress but define ω as the loss in value over face value of the contract to debtholders if default does occur. They obtain closed form solutions for the pricing of risky debt and also extend the methodology to swaps (in the working paper) but with some difficulties that are linked to the specific treatment of swaps. COSSIN and PIROTTE (1997b) go into the details of how to implement the LONGSTAFF and SCHWARTZ model and test the model with little success. As in

JONES, MASON, and ROSENFELD (1984), variables'construction is problematic.

Swaps are a complex instrument to analyze as far as credit risk is concerned, specially because they entail the credit risk of two counterparties. Classical models (COOPER and MELLO (1991) but also LONGSTAFF and SCHWARTZ (1995)) have not been fully dealt with that difficulty yet. See COSSIN and PIROTTE (1997a) for more on swap credit risk and for an empirical investigation of the problem.

The most interesting (although complex) developments in the theoretical field of credit risk may be in the combination of strategic debt servicing and CCA analysis. The treatment of the bankruptcy procedure is indeed by far too simple in the CCA models analyzed up until now and are not a realistic representation of the lengthy and complex negotiations that occur during financial distress. Different recent approaches try to modelize the strategic gaming that occurs and combine it to the CCA analysis (see for example ANDERSON and SUNDARESAN (1996), LELAND and TOFT (1996), MELLA-BARRAL and PERRAUDIN (1997)). These models give strong theoretical insights although they are still far from practical use. They have the theoretical advantage of modelizing clearly the full endogeneity of the bankrutpcy process and its impact on instrument pricing.

6. Alternative Models: A Mixed Approach

A more recent line of research takes a quite different approach. Although it still focuses on arbitrage free models, (and thus differentiates itself from the plain actuarial results of section 1), it gives up on endogeneizing the bankruptcy process in itself and considers it as an exogenous process. From a theoretical point of view, this is not a welcome concession. On the other hand, it allows for an easier treatment of practical cases (with the weakness of ignoring the financial economics behind the determination of the bankruptcy process). Many papers have recently appeared that follow this underlying assumption. I briefly present here only the few that seem the basis for the others'extensions.

LONGSTAFF and SCHWARTZ (1995) for example price simple credit derivatives with the strong assumption that the logarithm of the credit spread follows a mean reverting prespecified process.

JARROW and TURNBULL (1995) present a multinomial model working through a Forex analogy where the bankruptcy process is compared to a spot exchange rate process, instead of the previous papers' approach of default time being deducted from the process of the firm's value (or a related process). A bootstrap procedure allows to determine the martingale probabilities for default. It is used for European option valuation with a risky counterparty and for vulnerable option valuation with a risky writer. They also show the limiting case to be the Gaussian-Poisson model.

LANDO (1994) uses COX processes to model prices of credit risky bonds. In the line of JARROW and TURNBULL (1995), the event of default is not described as a function of the value of the firm. Default is presented as an unpredictable, Poisson-type event. The use of this type of model may prove challenging, as Poisson type events have been shown difficult to precisely estimate (high sensitivity to criteria of differences between continuous and discrete events).

By using an exogenous default process too, DUFFIE and HUANG (1996) formalize a model of swap pricing with two-sided default, one of the rare models to do so, by extending the approach used in DUFFIE and SINGLETON (1996). They are also able to study the impact of netting on the value of a swap portfolio when two-sided credit risk is involved. They develop numerical examples on both interest rate and currency swaps. Their model is the only one I know of that allows for actual, path-dependent pricing of two sided default, a major complication that arises in swaps (and it may be good to remind the reader that swaps are the most used derivatives instrument by very far, and that their credit risk is currently poorly taken into account). But their model also has the major weakness of relying on an exogenous bankruptcy process (rather than the endogenous process of MERTON (1974) and its followers). Although it makes the technique quite useable in practice, it also makes one doubt the validity of the calculations in the long run.

7. Conclusions

This paper exposed the current state of the art in credit risk valuation methods. Elements of applications to real world cases and how to build the necessary variables were discussed.

Option pricing theory (with the specific extensions to the topic) is a powerful tool to use, notably to value fixed income instruments with credit risk. Empirically, the analysis works best for simple structures (as in mortgage backed securities). But advances in the field are being made every day. The most significant recent advances on the theoretical side have been in the models combining interest rate risk and credit risk. As far as the short term practical use is concerned, models with exogenous bankruptcy process will surely attract the practitioners' attention. These models prove to be interesting when they handle situations that models with endogenous bankruptcy procedures cannot deal with easily (such as the double sided credit risk in swap contracts). Significant theoretical advances still need to be made in order to integrate endogenous bankrutpcy with realistic bankrutpcy proceedings.

But what maybe the most striking weakness of the current research, when looking at the rich academic theoretical developments appearing frequently, is the quasi absence of empirical research done on this topic.

Hopefully, the paper will show that there is much more to do for banks and other players in the field than set up credit lines (with no rational idea of the level to set them at). Credit risk is becoming an area of specialization in risk management that is extremely sophisticated, an area where arbitrage opportunities are still widely open. It should be obvious that every player will need to become more and more sophisticated regarding credit risk and that the most advanced players will enjoy a competitive advantage in the medium term and could very well reap nice profits from it.

Footnotes

[1] The formula can acommodate lump sum coupons of C_i at time t_i just by replacing c in the PDE by $\sum_i C_i \delta(t - t_i)$ where δ is the Dirac delta function.

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