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Efficiency of Long term Investment Strategies with Equity-Linked Notes

1. Introduction

One question that has been rarely addressed in financial literature is whether or not structured derivative products may be utilised in long-term investment strategies. After the Federal Reserve Board raised interest rates in February 1994 and the bear market began in 1994, quite a few investors got caught by a rather unpleasant surprise. Some structured derivative products in their portfolio 'turned sour' and as a result a lot of investors lost confidence in these financial instruments; a remarkable decline of the market for structured products could be observed until the end of 1994.

In this study we focus our interest in particular on both short- and long-term investment strategies with equity-linked notes (also known as synthetic equity, IGLU, GROI, PIP, Capital Protected Note). These are probably the most transparent structured derivative products. A typical equity-linked note consists essentially of a fixed-income instrument (such as a zero bond) and a European-style call option on some equity index. This financial instrument has not only the advantage to guarantee a certain minimum capital repayment at maturity but also allows the participation in the underlying equity market. At this point, however, an investor will ask about the efficiency of such a structured product as compared with a plain equity & money market investment. How are the risks in comparison to the offered rewards? This question as well as papers by BOOKSTABER/CLARKE (1984,1985), HARLOW (1991) and an editorial in this journal by ZIMMERMANN (1994) on different approaches to define and measure risks in the framework of modern portfolio theory have been the main motivation of our paper.

By put-call parity the analysis of investment strategies with equity-linked notes is very similar to the analysis of protective put strategies. The main difference is, however, that using put-call parity an equity-linked note is equivalent to holding stock, a put option and an additional amount of cash. Therefore our analysis is different from recent contributions of various academic researchers dealing with the efficiency of portfolio strategies with embedded options. We shall briefly summarize the main achievements of recent papers concerned with this topic and emphasize the differences to our own work.

- FERGUSON (1993) considers a one-period investment and provides some analytical formulae for expected returns and their variances for protective put and written call strategies. His formulae are derived under the assumption of lognormal stock price distributions and cannot

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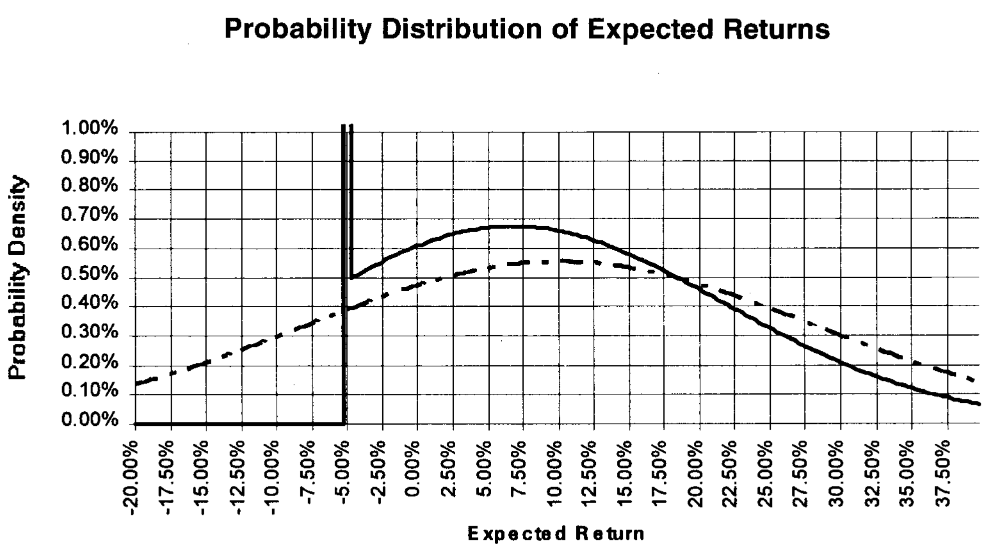
be applied to equity linked notes, since these are equivalent to stock, put option and an additional amount of cash.

- FIGLEWSKI et al. (1993) have used stochastic simulations in the Black-Scholes world (lognormal stock price process) to study different rolling put-strategies varying the market parameters (such as growth rate, volatility, etc.). In addition to stochastic simulations we present here some analytical arguments which allow us to construct efficiency curves for the one-period and the multi-period investment strategy in equity-linked notes. Especially asymptotic return distributions for multi-period equity-linked note strategies are obtained with our methods.
- MARMER/NG (1993) have discussed the role of synthetic equity in the context of portfolio optimisation for one period. Whereas they present efficiency diagrams in the one-period case for an efficient portfolio including bonds, equities and one synthetic equity, we compare the efficiency of one synthetic equity alone with a portfolio consisting of cash and equity for a downside risk measure in the one- and multi-period case.
- Finally we should like to mention a recent article of ALBRECHT et al (1995) in this journal. Their main contribution to this topic consists of a historical simulation of various rolling put-strategies using the price history of the DAFOX. In our case the historical simulation of equity-linked notes using the DAX history from 1975 to 1995 was of complementary character as we were interested to see how stock crashes, changing interest rates, changing volatilities etc. modify our theoretical findings.

The aim of this paper is to continue the discussion of risk-measures as presented in the above publications and to apply these concepts to structured products such as equity-linked notes.
For our empirical part we have choosen the DAX (Deutscher Aktienindex) as the underlying stock market index because it has a sufficient long history and it is a performance index. Therefore all

Figure 1: Probability distribution density of continuously compounded expected returns for 95%-IGLU (bold line) and index (dashed line) for market scenario with $r = 5\%$ p.a., $\mu = 10\%$ p.a., $\sigma = 15\%$p.a. and $T = 1y$



dividends are reinvested and thus one does not need to make any assumptions on the size of the dividend payments nor whether dividends are paid continuously or only at certain dates. This simplifies the option valuation significantly. The empirical study uses data from the time interval July 1975–March 1995.

Our study is structured as follows. Section 2 gives a brief overview of the analytical results that can be obtained under the assumption of a lognormal stock market model. Section 3 offers a more in-depth analysis of the risk/return characteristics of IGLUs in the one and multi-period case. In section 4 we present the results of the empirical analysis. We conclude with a discussion of the findings in section 5.

2. Theoretical Analysis

In this section we want to present briefly the theoretical framework of our study. The analysis will be completed by a Monte-Carlo-simulation for the cases for which it was not possible to find a closed form solution. The obvious advantage of a theoretical analysis is that a broad range of results can be obtained. Specifically, we shall vary the maturity of the IGLUs, the capital protection level, the length of the investment horizon, the level of interest rates and volatility.

We divide our analysis into two parts. The first one, the one period case, refers to the study of the characteristics of an IGLU for the period until ist expiration: the investment period, T, coincides with the maturity of the IGLU, τ. The second part, the multi-period case, refers to the instance in which the investment horizon is longer than the life of a single IGLU, thus implying a rollover IGLU strategy.

2.1 The one-period case: $\tau = T$

Our theoretical assumptions are the conventional ones in finance (e.g. HULL (1993)): the stock price S_t at time t is described by a lognormal process with constant growth rate μ, constant volatility σ, and constant risk-free rate r and no dividend payments,

$$S_t = S_0 \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right) \cdot t + \sigma \cdot W_t} \tag{1}$$

where W_t denotes the standard Brownian motion at time t. In the one-period analysis one readily obtains the expression for the expected return $\bar{y}^{(T)}$ per annum for the investment period T on the stock

$$\bar{y}^{(T)} = E_0\left(\frac{1}{T} \cdot \ln(\frac{S_T}{S_0})\right) \tag{2}$$

where the expectation $E_0(\cdot)$ at time $t = 0$ can be calculated explicitly (see Appendix A for more details).

The equity-linked note, henceforth named for simplicity IGLU (Index Growth Linked Unit) is a structured product consisting of a *zero bond* to cover the capital guarantee and a *call option* on some stock. For a capital guarantee of κ, where $0 < \kappa \le 1$, the strike of the call option is related to this number by setting $K = \kappa \cdot S_0$. Thus the call option is in-the-money. Therefore if an amount I_0 is invested at time $t = 0$ in equity-linked notes, the value of such an investment at expiry date T ($= \tau$) is given by

$$I_T = \kappa \cdot I_0 + \max\left\{0, \frac{S_T}{S_0} - \kappa\right\} \cdot I_0 \cdot \frac{\left(1 - \kappa \cdot e^{-r \cdot T}\right)}{\mathrm{call}(\kappa, T, r, \sigma)} \tag{3}$$

where $\left(1 - \kappa \cdot e^{-r \cdot T}\right) \cdot I_0$ denotes the amount of money available to buy call-options and $\mathrm{call}(\kappa, T, r, \sigma)$ denotes the Black-Scholes price for a call option on the quotient S_T / S_0 with strike κ and time to expiry T. We can then define the expected return for the one-period investment of length $T = \tau$, $\bar{Y}^{(T)}$, in a similar fashion as above

$$\bar{Y}^{(T)} = E_0\left(\frac{1}{T} \cdot \ln(\frac{I_T}{I_0})\right) \tag{4}$$

Table 1: Changing the expected return of the underlying index

Expected Index Return (p.a.)	70%-IGLU	80%-IGLU	90%-IGLU	95%-IGLU	100%-IGLU
0%	0%	0.08%	1.04%	2.36%	4.44%
5%	5%	5.02%	5.23%	5.52%	5.89%
10%	10%	9.98%.	9.64%	9.00%	7.60%
15%	15%	14.96%	14.21%	12.75%	9.53%
20%	20%	19.94%	18.89%	16.71%	11.68%

For high expected returns of the underlying index we see that the expected return of the IGLU decreases with increasing capital protection because the „protection costs“ become dearer. For low expected returns of the underlying index the IGLU-expected return increases and is even higher than the return of the underlying index. A high floor protects the IGLU from negative returns, boosting the expected return and offsetting the protection costs.

Table 2: Changing volatility

Volatility (p.a.)	Expected Index Return (p.a.)	70%-IGLU	80%-IGLU	90%-IGLU	95%-IGLU	100%-IGLU
5.00%	**7.50%**	7.50%	7.50%	7.50%	7.49%	7.12%
10.00%	**7.50%**	7.50%	7.50%	7.47%	7.32%	6.79%
15.00%	**7.50%**	7.50%	7.50%	7.41%	7.22%	6.71%
20.00%	**7.50%**	7.50%	7.50%	7.44%	7.26%	6.74%
25.00%	**7.50%**	7.51%	7.56%	7.56%	7.39%	6.81%

Similar to Table 1 the protection costs cause the IGLU expected returns to be slightly lower than the returns of the underlying index. Only at high volatilities (and as a confirmation of the empirical study in Section 4) the 80% to 90%-IGLUs offer the best combination of low protection cost and effective protection. Their expected returns are even higher than the underlying index's.

where the expectation $E_0(\cdot)$ at time $t = 0$ is calculated by numerical integration (see [1], and Appendix B).
Note also that in the Black-Scholes world we can easily obtain expressions for the probabability distribution for the returns of index and IGLU.The graph clearly indicates that an IGLU is an equity investment with limited downside risk.

In tables 1 to 4 we demonstrate how the expected return of an IGLU held until maturity changes by varying the main input parameters. We set the volatility of the underlying equity index at 15% p.a., its expected return at 7.50% p.a. and its maturity to 0.5 years. The money market rate is 6% p.a.. In the tables we change each parameter separately, holding the remaining parameters unchanged.

2.2 The multi-period analysis: $\tau < T$

Suppose we follow now an investment strategy in which some initial amount is invested in equity-linked notes and at maturity of these equity-linked notes the proceeds are re-invested in new equity-linked notes with the same maturity and capital protection. In this multi-period case we denote by n the number of investment periods so that T is equal to $n \cdot \tau$. We define the annualised continuously compounded return over n periods (see Appendix B) as:

$$y^{(T)} = \frac{y_1^{(\tau)} + y_2^{(\tau)} + \cdots + y_n^{(\tau)}}{n} \qquad (5)$$

where $y_i^{(\tau)}$, n, denotes the continuously compounded return in the i-th period. This holds for both the stock portfolio and the IGLUs. By some arguments briefly outlined in Appendix A we

Table 3: Changing the interest rate

Interest Rate (p.a.)	Expected Index Return (p.a.)	70%-IGLU	80%-IGLU	90%-IGLU	95%-IGLU	100%-IGLU
4%	**7.50%**	7.50%	7.48%	7.12%	6.47%	5.03%
6%	**7.50%**	7.50%	7.50%	7.41%	7.22%	6.71%
8%	**7.50%**	7.50%	7.51%	7.64%	7.81%	8.03%
10%	**7.50%**	7.50%	7.52%	7.81%	8.27%	9.05%
12%	**7.50%**	7.50%	7.53%	7.95%	8.64%	9.85%

High interest rates lower the price of the zero-bond and thus the protection costs for a given floor-level decreases. At the same time the call price increases, but not as much as the zero bond price decreases. This results in IGLU expected returns which are higher than the return of the underlying index.

Table 4: Changing the IGLU-maturity

Maturity	Expected Index Return (p.a.)	70%-IGLU	80%-IGLU	90%-IGLU	95%-IGLU	100%-IGLU
1 month	**7.50%**	7.50%	7.50%	7.50%	7.40%	6.32%
3 months	**7.50%**	7.50%	7.50%	7.46%	7.28%	6.54%
6 months	**7.50%**	7.50%	7.50%	7.41%	7.22%	6.71%
12 months	**7.50%**	7.50%	7.48%	7.37%	7.22%	6.91%
24 months	**7.50%**	7.49%	7.46%	7.36%	7.27%	7.10%

Longer maturities make the price of the embedded zero-bond and therefore of the protection cheaper but also increase the price of the embedded option. The zero bond effect, however, is greater than the option effect. This favours long dated high-protection IGLUs.

know that the random variables $y_i^{(\tau)}$, $i = 1,\dots,n$, are independent identically distributed random variables with equal mean and variances,

$$E_0\left(y_i^{(\tau)}\right) = \bar{y}^{(\tau)}, \quad \mathrm{var}\left(y_i^{(\tau)}\right) = \sigma^2 \tag{6}$$

and finite third moment. We may therefore apply the central limit theorem on the random sum for $y^{(T)} = y^{(n*\tau)}$ if n gets large. Thus we conclude that $y^{(T)}$ is normally distributed with mean $\bar{y}^{(\tau)}$ and variance σ^2/n in the limit $n \to \infty$ [2]. To appreciate this non-trivial observation one may recall that the distribution of the IGLU returns is not normal and very asymmetric. Figures 2 and 3, which were obtained by Monte Carlo simulations[3], demonstrate how the asymmetric return distribution of the IGLU from Figure 1 becomes more and more „normal" as n gets bigger, confirming the theoretical argument from above. Therefore the standard deviation becomes a sensible risk measure in this limit again. Note also that the risk not to achieve the expected average return, $\bar{y}^{(\tau)}$, over n investment periods vanishes as $n \to \infty$.

This last result is intuitively understandable for a stock-portfolio whose instantaneous returns are normally distributed with constant variance and drift. Indeed, there is no difference between a single- and a multi-period strategy. In a world without transaction costs (as in our analysis) it is equivalent to roll-over the same stock-portfolio every τ years for n times (sell the portfolio and buy it just afterwards) or to keep the portfolio without transactions until T.

Figure 2: Long-term return distribution of a 100% IGLU (thin line) and the index (bold line) for 1y investments with 5 roll-overs

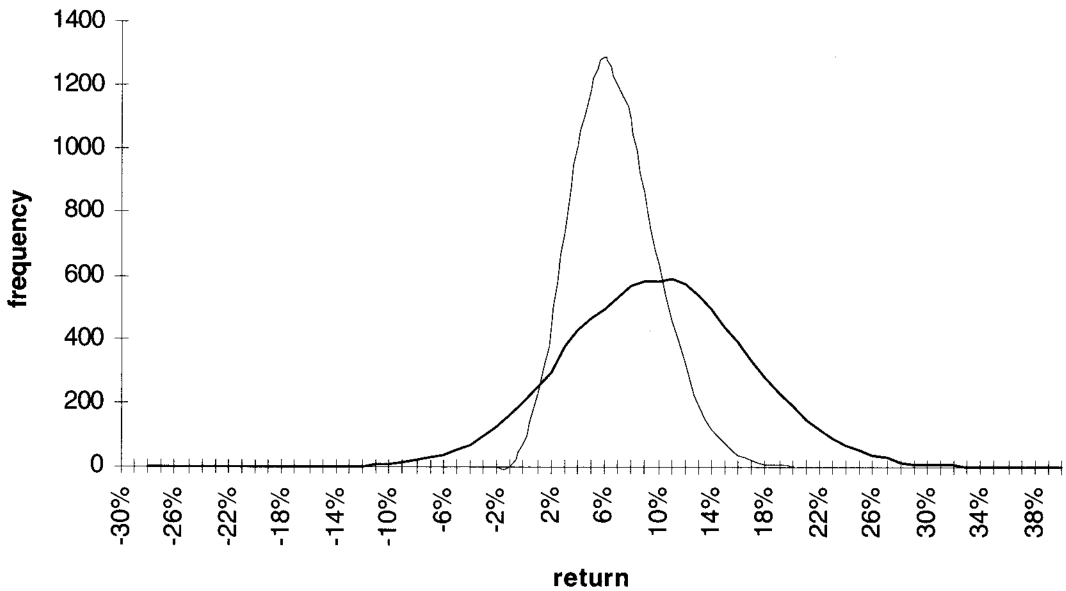
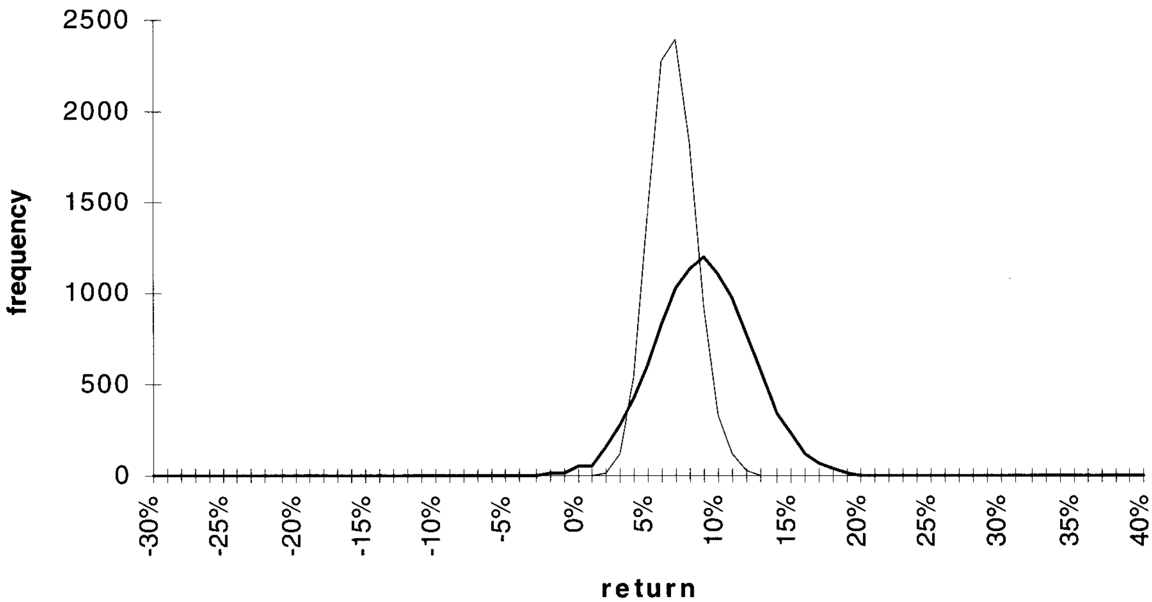


Figure 3: Long-term return distribution of a 100% IGLU (thin line) and the index (bold line) for 1y investments with 20 roll-overs. Note that the long-term return distribution of the IGLU looks „normal"



3. Efficiency of Equity-linked notes versus Money Market & Equity

A more appropriate judgement of the efficiency of equity-linked notes versus cash & equity portfolios requires according to MARKOWITZ (1959) the comparison of the expected returns and corresponding risks. However, the traditional risk measure, the volatility of the returns, cannot be used in the context of strongly asymmetric return distributions (see Figure 1 for one period and Figures 2 and 3 for several holding periods); therefore, alternative risk measures should be considered.

Our main motivation for implementing an IGLU investment strategy is our interest in downside protection. Thus the potential risk-measures should be those which emphasize the risk of being worse than a certain target return. Downside risk mea-sures have been proposed firstly by ROY (1952) in an almost forgotten paper with rather general findings. The idea was later taken up bei LEIBOWITZ and other co-workers in several papers (e.g. LEIBOWITZ and LANGETIEG (1989)). For some recent references on this topic HARLOW (1991), ZIMMERMANN (1994), and ALBRECHT et al. (1995) may be used.

Although we have investigated each of the downside risk measures suggested in the above references[4], we have chosen the conditional semivariance (i.e. second lower partial moment with threshold equal expected return) as downside risk measure for two reasons. Firstly, this risk-measure becomes the standard volatility for normal distributions. Secondly, portfolio strategies are always concerned about the variability of not achieving the target return, which is equal to the expected return. The conditional semivariance is formally defined as follow

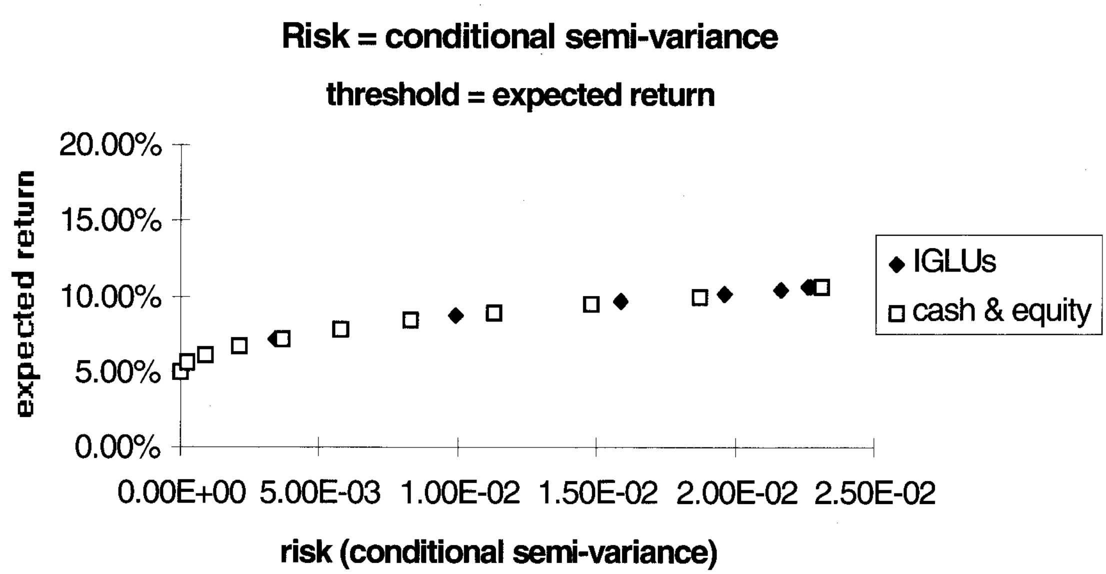
$$\mathrm{CSV}(\bar{y}) \quad = \quad E_0\left([y-\bar{y}]^2 \,|\, y<\bar{y}\right) \qquad (7)$$

where the threshold return $\bar{y} = E(y)$ is equal to the expected return.
After defining the risk-measure we shall compare the risk/return characteristics of various IGLUs with cash & equity investment portfolios. By varying the cash and equity composition of this benchmark portfolio it is possible to achieve different risk/return combinations and therefore define a „benchmark risk/return frontier"[5]. Note, however, that the problem of the optimal portfolio is beyond the scope of this investigation.

3.1. Risk/Return Characteristics of IGLUs in the one-period case

We see in Figure 4 that for a one period investment IGLUs with different capital protections are located on the efficiency curve defined by benchmark portfolios containing cash & equity. Keeping in mind that an equity call option at a certain time is equivalent to a certain long position in stock and a short position in cash, the IGLU thus corresponds to a portfolio consisting of cash & equity in certain proportions. Thus the term synthetic equity seems to be appropriate for IGLUs. Similar diagrams are obtained if maturity, interest rate, volatility and growth rate of the underlying stock are varied.
Since equity returns are normally distributed and the conditional semivariance therefore equal the variance, the cash & equity portfolio corresponds to the well-known capital market line. If one had choosen the square root of the conditional semivariance as risk measure (i.e. the standard deviation), one would obtain the usual straight capital market line.

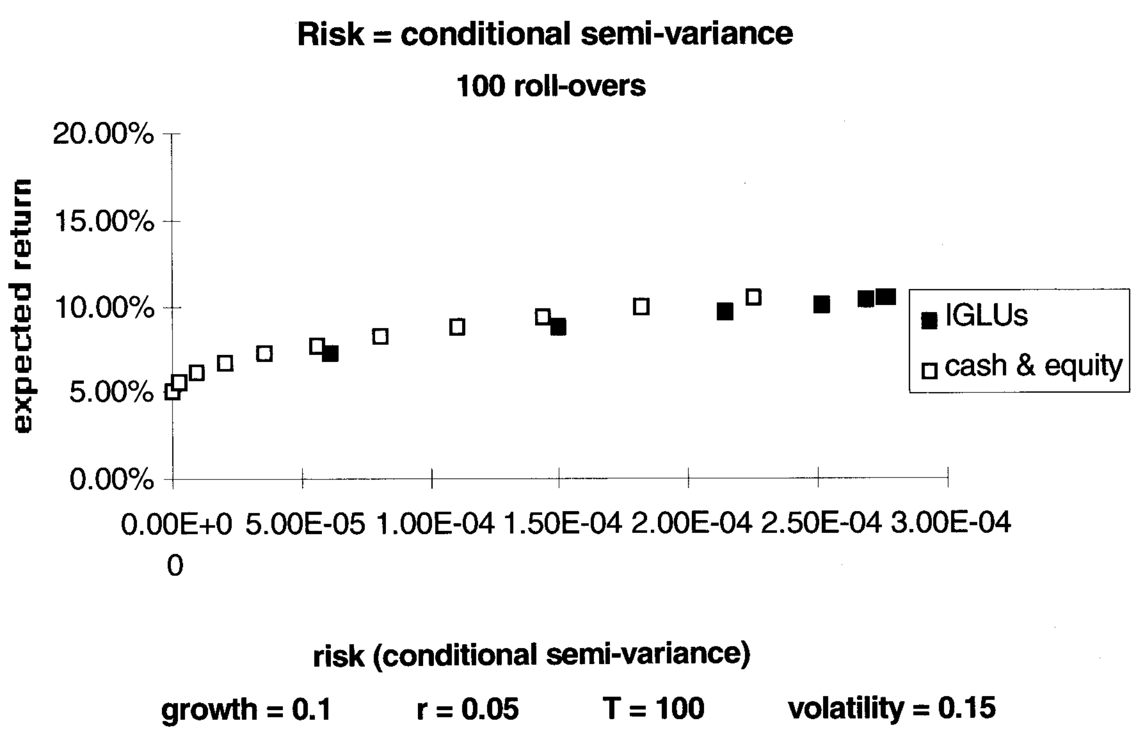
Figure 4: Risk/expected return curve. Risk = conditional semivariance, threshold = E(y). The white squares correspond to portfolios with α% equity in ascending order from 0% to 100% with 10% steps; The black squares correspond to IGLUs with the following floors: 100%, 95%, 85%, 80%, 75%, 70%



3.2. Risk/Return Characteristics of IGLUs in the multi-period case

Analogue to section 2.2 we consider long term strategies with multiple rollovers. Figure 5 indicates that the IGLU strategy is only slightly worse than the benchmark cash & equity portfolios. Similar diagrams are obtained if maturity, interest rate, volatility and growth rate of the underlying stock are varied. Figure 5 looks similar to the results on protective put strategies over one investment period presented by BOOKSTABER/CLARKE (1983, 1984, 1985), who measured the risk by the traditional variance. To explain this similarity, recall that as the number of rollovers increases the distribution of returns tends to a normal distribution (see section 2.2 and Figure 3). Therefore the conditional variance is also very close to the variance and we may compare our results for IGLUs, which are by put-call paritiy equivalent to a position consisting of cash, equity and a put-option, with the protective put strategies of BOOKSTABER/CLARKE (1983, 1984, 1985). Furthermore note that the path to achieve the expected long-term return is much smoother with IGLUs because of their downside protection. Our risk measure does not, however, take into account this positive feature.

Figure 5: Risk/expected return curve. Risk = conditional semi-variance, threshold = E(y), n = 100, τ = 1 y. The white squares correspond to portfolios with α% equity in ascending order from 0% to 100% with 10% steps; The black squares correspond to IGLUs with the following floors: 100%, 95%, 85%, 80%, 75%, 70%



4. Results of the Empirical analysis on the DAX

In the historical study we investigate how the long-term investment strategy described in section 2.2 using equity-linked notes performs in comparison with a direct investment in the DAX. We assume that our DAX-IGLUs with maturities of 6 months are perfectly divisible and that there are neither bid/ask spreads nor transaction costs. Since the cumulative returns depend on the particular path we investigate for the sake of statistics 6 different paths, starting on the first business day of each calendar month. Thus we have a strategy starting in January, in February, and so on. At the end of each holding period the return of the IGLU is annualized and compared with the 6-month return p.a. achieved by the DAX itself. Figure 6 displays the distribution of the returns for one holding-period (see also ALBRECHT et al. (1995)).

The characteristics of the IGLUs vary according to the historical market conditions prevailing at the relevant dates. Thus in periods with high interest rates and low volatility, for example, the participation factor was quite high because the price of the embedded call-options on the DAX was low and the price of the zero-bond was also low. As in ALBRECHT (1995) historical volatilities have been used as mid implied volatilities. To account for the fact that there is usually a bid-ask spread for implied volatilities we increment the call option volatility by 1 percent point. The historical volatilities have been calculated from the daily returns of the six month period before the launch of an IGLU (see e.g. HULL (1993)).

The results of one particular strategy are displayed in Figure 7. They give a flavour of how the IGLU-strategy provides protection if the underlying index crashes.

Figure 8 indicates that even though one had to forego upside potential (participation is always less than 100%) it was nevertheless possible to achieve a long-term return very close to the stock market.

Figure 6: Return distribution of the DAX and the IGLU strategy: period from July '75 to March '95

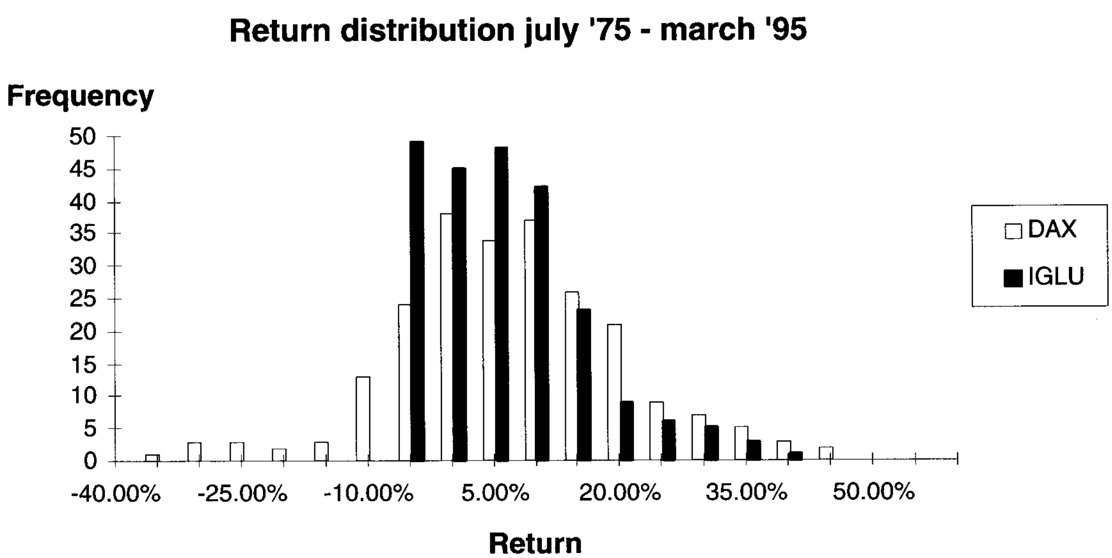


Figure 7: 6-month returns of the 95% IGLU and DAX: August path (i.e. 6-month investment strategy in 95%-IGLUs starting on the first business day in August 1975)

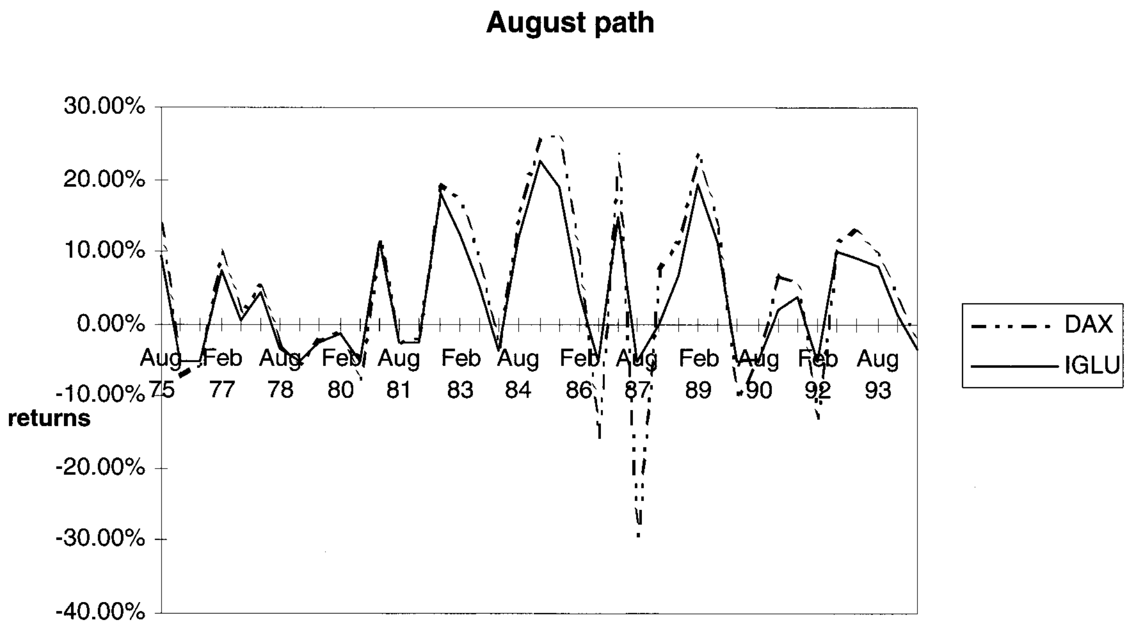


Figure 8: Value of investment for the 95%-IGLU and the DAX: August path (i.e. 6-month investment strategy in 95%-IGLUs starting on the first business day in August 1975)

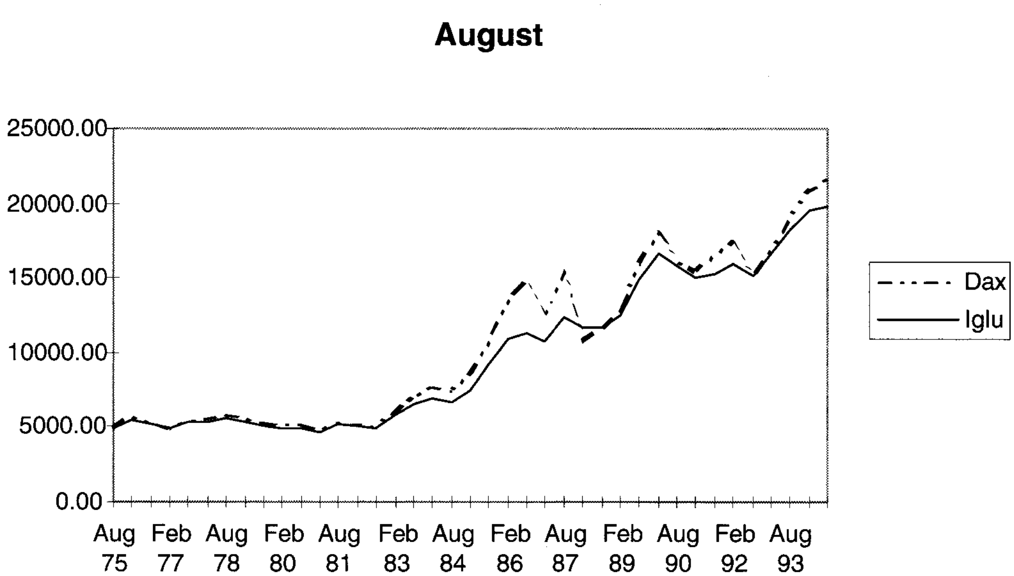


Table 5: Average return p.a. of DAX and IGLUs with different capital protection. Note that the 80%- and 85%-IGLUs outperform the market

DAX	50%-IGLU	60%-IGLU	70%-IGLU	**80%-IGLU**	**85%-IGLU**	90%-IGLU	95%-IGLU	100%-IGLU
7.48%.	7.47%	7.43%	7.38%	**7.69%**	**7.67%**	7.47%	7.35%	7.03%

As expected we see in Table 5 that the IGLU returns decrease with increasing capital protection. However, at the 80%-level they rise abruptly (they are even higher than the DAX-return) and then decline again. This is due to the fact that 80% represents apparently an optimal combination of capital protection and participation. Indeed, the 80% and the 85% IGLU-strategies were able to absorb the impact of strong negative market phases (such as the October 1987 stock market crash) without costing too much. A capital protection below 80% was cheap but useless. On the other hand a protection above 85% did absorb the shocks but was expensive.

We would like to stress the fact that these numbers result from the unique history of the DAX over the last 20 years. A discrepancy, however, to our theoretical analysis is that the IGLUs' expected returns decrease monotonically with a higher floor. This is due to the fact that the assumption of normally distributed equity returns excludes the occurrence of crashes in the theoretical case. Only for very high volatilities do we see a similar return-pattern.

5. Conclusions

Our conclusions including findings from both the single-period and multi-period analysis are summarised in this section. The main results of the one-period analysis may be summarised as follows:

- Usually IGLU expected returns are only slightly lower than the expected return of the underlying index. In some cases they are even higher.
- High-protection IGLUs have a very attractive expected return relative to the underlying index when interest rates are high and when the return expectation for the market is low.
- In highly volatile markets the protection level may be reduced to 80% .
- The higher the protection, the longer the maturity of the IGLU should be chosen.
- Even in the multi-period case the expected returns of IGLUs are only slightly lower than the expected returns of the underlying index.
- In the multi-period case IGLU-returns are significantly less volatile than the returns of the underlying index. In multi-period IGLU-strategies the likelihood to achieve the expected return is much more certain in comparison with the underlying index.
- Due to the higher certainty of achieving the expected return over more periods, IGLUs are equity investments which guarantee „peace of mind" and are not too dear.

Appendix 1:

Calculation of Expected Returns

The expectation $E_0(\cdot)$ used in the previous sections is not to be confused with the risk-neutral expectation but defined as

$$E_0(\cdot) = \int_{-\infty}^{+\infty} \cdot \; p(x)\,dx$$

where $x = \ln(S_T / S_0)$ and $p(x)$ denotes the normal probability density given by

$$p(x) = \frac{1}{\sqrt{2\pi\,\sigma^2 T}} \cdot e^{-\frac{\left[x-\left(\mu-\sigma^2/2\right)\cdot T\right]^2}{2\cdot\sigma^2 T}}$$

Analytic calculation of the relevant integrals reveals the follwing expression for the expected continuously componded return, $\overline{y}^{(T)}$,

$$\overline{y}^{(T)} = E_0\left(\frac{1}{T}\cdot\ln\frac{S_T}{S_0}\right) = \mu - \frac{\sigma^2}{2}$$

For the IGLU the expected continously compounded returns, $\overline{Y}^{(T)}$, is obtained by evaluating the following defining expression

$$\overline{Y}^{(T)} = E_0\left(\frac{1}{T}\cdot\ln\frac{I_T}{I_0}\right)$$

$$= \frac{1}{T}\cdot E_0\left(\ln\left[\kappa + \max\left\{0, \frac{S_T}{S_0} - \kappa\right\}\cdot\frac{\left(1-\kappa\cdot e^{-r\cdot T}\right)}{\text{call}(\kappa,T,r,\sigma)}\right]\right)$$

The expectation for the calculation of $\overline{Y}^{(T)}$ needs to be evaluated e.g. by numerical integration. Finally note that due to the properties of the lognormal stock price process and our assumptions from above the following identities hold

$$E_0\left(\ln\frac{S_{n\cdot T}}{S_{(n-1)\cdot T}}\right) = E_0\left(\ln\frac{S_T}{S_0}\right)$$

$$E_0\left(\frac{S_{n\cdot T}}{S_{(n-1)\cdot T}}\right) = E_0\left(\frac{S_T}{S_0}\right)$$

where the integer n denotes the number of the investment period which starts at time $(n-1)\cdot T$ and ends at time $n\cdot T$. In other words, the returns on the stock depend only on the length of the investment period, T, but not on the actual date when the investment period starts. By the same arguments we conclude that this is also true for the IGLU investment strategy

$$E_0\left(\ln\frac{I_{n\cdot T}}{I_{(n-1)\cdot T}}\right) = E_0\left(\ln\frac{I_T}{I_0}\right)$$

$$E_0\left(\frac{I_{n\cdot T}}{I_{(n-1)\cdot T}}\right) = E_0\left(\frac{I_T}{I_0}\right)$$

Due to this property we conclude that the continuously compounded returns, $\overline{y}^{(T)}$ and $\overline{Y}^{(T)}$, are equal for each time period and equal the total return of the iterated investment strategy proposed in section 2.

Appendix 2:

Calculation of Probability Distributions of Returns

Some elementary probabilistic arguments lead to the normal distribution of the continuous return on the stock index, y,

$$p(y) = \frac{1}{\sqrt{2\pi \cdot \frac{\sigma^2}{T}}} \cdot e^{-\frac{\left[y-\left(\mu-\sigma^2/2\right)\right]^2}{2\cdot\frac{\sigma^2}{T}}}$$

The probability distribution for the continuous returns on the IGLU is derived by similar probabilistic arguments: After some rather tedious calculations one obtains an expression for the probability, that the actual IGLU return is bigger than some real number Y. Differentiation of this probability with respect to Y yields the desired probability density for the IGLU returns

$$p(Y) = \delta\left(Y-\frac{1}{T}\ln\kappa\right)\cdot N\left(\frac{\ln\kappa-\left(\mu-\sigma^2/2\right)\cdot T}{\sigma\cdot\sqrt{T}}\right)$$
$$+ 1_{Y>\frac{1}{T}\cdot\ln\kappa} \cdot \frac{1}{\sqrt{2\pi\cdot\sigma^2/T}}\cdot e^{-\frac{\left[x-\left(\mu-\sigma^2/2\right)\cdot T\right]^2}{2\sigma^2\cdot T}}\Bigg|_{x=\ln\left(\frac{e^{Y\cdot t}-\kappa}{A}+\kappa\right)}$$

Note that there is a δ-peak at the minimum return of $\frac{1}{T}\ln\kappa$.

Furthermore, for returns lower than this minimum value the probability density vanishes identically.

Our derived probability distributions are used for the numerical calculation of expected returns and semivariances. The graphs of the probability distribution functions are depicted in Figure 1. This figure may be compared with Figure 2 in BOOKSTABER/CLARKE (1984) for fully covered portfolios (i.e. the equity exposure is entirely protected by put options). Note however, that although these authors make similar theoretical assumptions on the stock returns, their problem of optioned-stock portfolios is more general than ours: their degree of protection may vary from 0% to 100% and more than one stock is taken into consideration. Therefore an explicit representation of a probability density function is not feasable in this case.

Footnotes

[1] FERGUSON (1993) is able to calculate various quantities explicitely since he arrives at expressions like $E_0\left(\ln\left(K + \max\{S_T - K,0\}\right)\right) = E_0\left(\ln\left(S_T + \max\{K - S_T\}\right)\right)$ which can be simplified to $E_0\left(\ln(S_T)\cdot 1_{S_T>K}\right) + E_0\left(\ln(K)\cdot 1_{S_T\leq K}\right)$.
These last expressions can be evaluated analytically. In our case we derive expressions like
$E_0\left(\ln\left(K + \alpha\cdot\max\{S_T - K,0\}\right)\right)$,
where $0 < \alpha < 1$ denotes the participation. FERGUSON's trick cannot be applied here and the integrals have to be solved numerically.

[2] For stock returns this result is obvious since the period returns are normally distributed and the sum of normal distributions yields a normal distribution again.

[3] In the Monte Carlo simulations samples of 100'000 random returns were generated for each set of market parameters. From these sample sets we generated 10'000 random subsamples of various length, n = 5, 100, to compute the annualised continuously compounded return over n periods,
$y^{(T)} = y^{(n*\tau)}$.
Figures 2 and 3 display the probability distribution obtained from smoothed histograms of these data.

[4] For example the main advantage of the shortfall probability, i.e. the probability that the return is lower than a pre-defined threshold return, is that it is easy to calculate and to understand; however it is not applicable to „extreme events", meaning: it does not measure how „far below" the threshold level the returns can occur. Assume for example that our threshold level is 0% and that two financial instruments are given. The first one yields with a probability of 10% a return of –5%otherwise a return of 10%. The second yields with the same probability a return of –50%-otherwise a return of 15%. The short-fall probability of the two instruments is the same (namely 10%) although the second instrument would appear to have more „risk" than the first one.

[5] The calculation of the various expectations is done numerically using the probability distributions in Appendix B for the one-period case in section 3.1. For the multi-period case in section 3.2 the expectations are calculated via Monte Carlo simulation. If the number of periods, n, is sufficiently large, the probability distribution may be approximated by a normal distribution (see also the results in section 2.2 and also Figure 3). In this case the diagrams obtained by Monte Carlo and this approximation are virtually the same.

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