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Hedging with Multiple Interest Rate Futures

1. Introduction

The increase in volatility of interest rates since the 1970s made low-risk bond funds subject to substantial price risk. When interest rates go up, the price of outstanding bonds declines to compensate for the constant coupon which remains unchanged. Thus investment vehicles such as well rated corporate bonds or Treasury issues which are intended to be low-risk securities, are risky investments in volatile interest rate markets. Interest rate futures contracts offer an important alternative for risk control. This paper presents several hedging methods including traditional hedging techniques and more recent results in a form suitable for use by money managers.

Interest rate futures are financial contracts which call for delivery of some basket of fixed-income securities to be delivered at an expiration date, for a previously specified futures price. There exist a number of important interest rate futures contracts which differ in the type and size of the delivery. Organized markets for interest rate futures exist in the United States, the United Kingdom, Japan, and other countries. Among the most liquid markets are the U.S. Treasury bill and Eurodollar deposits for short term interest rates, traded on the International Money Market, the U.S. Treasury bond and a contract on mortgage bonds known as GNMA, traded on the Chicago Board of Trade, the ten-year bond with a 6% yield traded on the Tokyo Stock Exchange, the three-month sterling deposit contract traded on the London International Futures Exchange and Eurodollar contracts traded on several exchanges in the United States, London, Canada and other countries. The two main problems faced by the bond portfolio manager are instrument adequacy and optimal hedge ratio. When the hedged cash position consists of securities which are deliverable under a futures contract, then it is possible to lock in a future trading price. This situation is called a perfect hedge and is important in a theoretical sense, as it facilitates the pedagogical explanation of hedging with interest rate futures. However, most often the cash position to be hedged is not deliverable under a futures contract and then cross hedging is the only alternative. Positions are assumed in futures contracts which are correlated with the spot hedged position. The residual risk is known as basis risk, as it pertains to the basis, or the difference between the futures price and the spot price. Selection of the most appropriate contract is important, as the hedging effectiveness relative to the spot position is contract specific.

This paper presents how theoretical hedging models can be used by money managers. Besides instrument adequacy, the optimal hedge ratio, or the optimal number of futures contracts to be as-sumed is of critical importance. A number of models for hedging with interest rate futures have been proposed in the academic literature and employed by practitioners. Some of these techniques are extensions from hedging theory for commodity futures such as the variance minimization, price regression, with single or multiple contracts. Other approaches specifically account for the characteristics of the bonds such as coupon, schedule of payments and maturity dates. These methods include the direct price sensitivity estimation technique, the duration approach, and the rate diffusion process approach. Early research by WORKING (1953), ANDERSON and DANTHINE (1980) and NELSON and COLLINS (1985) suggests that hedgers are not only risk minimizers but also seek high expected returns.

In this paper particular attention is given to the extension of the diffusion process rate for multiple futures. The empirical analysis section investigates the relative strengths and limitations of multiple versus single contract techniques. The models presented in the paper are generally applicable for any interest rate futures contract, and some of the methods are illustrated using two of the most actively traded futures, the U.S. T-bond and T-bill contracts.

The paper is organized as follows. Section 2 presents two equivalent hedging methods, the minimum variance and price regression. The yield sensitivity techniques by direct calculation and using the duration approach are presented in sections 3 and 4 respectively. The rate diffusion process approach is presented in section 5 and its direct implication to hedging with multiple futures is set forth in section 6. Comparative empirical hedging with single and multiple contracts is conducted in section 7 and section 8 concludes the paper.

2. The Minimum Variance Approach

Traditional hedging theory represents an extension of portfolio theory to hedging by emphasizing risk avoidance. Hedgers are individuals with a vested interest in a spot commodity who protect the value of their holding by taking an offseting position in the futures market. Losses or gains in the spot (cash) position are offset by gains or losses in the futures markets and considerable uncertainty is removed. However, there exists residual risk due to less than perfect correlation between the spot and futures position. Cross hedging with interest rate futures represents trading interest rate risk for basis risk.

The minimum variance model is a natural application of traditional price hedging with commodity futures. The method minimizes risk as measured by the variance of portfolio returns. Following the early works of JOHNSON (1960) and STEIN (1961), MAKIN (1978) and McENALLY and RICE (1979) derive minimum variance hedge ratios for commodity futures contracts. EDERINGTON (1979) extends the theory to hedging with GNMA and T-Bill interest rate futures.

Let P_0 denote the current price of a bond and $\tilde{P}_1$ be the random value of that bond at the end of the period over which risk minimization is desired. Similarly, denote by F_0 the invoice price of the deliverable instruments under the futures contract implied by the current futures price, and $\tilde{F}_1$ the invoice price at the end of the period. The futures invoice price represents the price of the delivery at the terms of the contract. For example the T-bond futures contract calls for delivery of T-bonds with a face value of \$100'000. The contract is quoted based on a hypothetical 8% coupon bond with 20 years to maturity, but any bond with fifteen years or longer to maturity is deliverable. The invoice price is adjusted to reflect the coupon and maturity of the bond being delivered. Occasionally, when confusion is not possible, we use the terms of futures price and invoice price interchangingly and denote it by F.

The change in bond value $\Delta P = \tilde{P}_1 - P_0$ and the change in futures price $\Delta F = \tilde{F}_1 - F_0$ are random. For a portfolio V consisting of the bond P and h futures contracts, the profit is equal to interest + $\Delta P + h\,\Delta F$. The uncertainty of the dollar return as measured by its variance is $var_v = var(\Delta P) + 2h cov(\Delta P, \Delta F) + h^2 var(\Delta F)$. The number h of contracts which minimizes the portfolio variance can be obtained by solving the equation $\delta var_v / \delta h = 0$, where δ denotes a partial derivative. The minimum variance futures position is:

$$h^* = - \frac{cov(\Delta P, \Delta F)}{var(\Delta F)} \tag{1}$$

If the variance and covariance of the quoted price rather than the invoice price are used, then h represents the hedge ratio, or the dollar value of the futures position per dollar of bond inventory. The application of the minimum variance method is straight forward. The hedge ratio h* coincides with the second coefficient in the regression of bond price changes with respect to futures price changes. Given a time series of bond and futures prices $(P_i, F_i)_{i=1,...,n}$ the minimum variance futures position can be derived as minus the coefficient of $F_{i+1}-F_i$ in the regression

$$P_{i+1} - P_i = \alpha + \beta(F_{i+1}-F_i) + \tilde{\varepsilon} \tag{2}$$

The minimum variance model is extended by HILLIARD (1984) to hedging with multiple contracts and it is widely used by practitioners because of its computational simplicity. The regression technique is intuitively appealing as it recommends the futures position which is most likely to offset the uncertain price change in the spot market.

The minimum variance method provides a useful tool for controlling the basis risk. However, unlike other non-financial commodities which maintain the same price sensitivity relative to their futures contracts, the regression coefficients of fixed-income instruments may not be stable. For example suppose that a two-year bond is hedged with T-bill futures and that the minimum variance hedge ratio is determined by price regression using three years of bond price history. Then, the estimated regression coefficients reflect the relative price sensitivity of a more sensitive bond whose time to maturity ranges between three and five years and not that of the two-year bond being hedged.

3. The Yield Sensitivity Technique by Direct Calculation

It is possible to anticipate the change in bond value in response to changes in interest rates by specifically accounting for the coupon and payment dates. Assuming parallel interest rate shifts, the yield sensitivity technique uses discounted cash flow analysis to estimate directly price changes in the futures and spot markets. The number of futures contracts h* which offsets the price variation in the spot position is $h^* = - \Delta P/\Delta F$.

Denote by r_B the yield to maturity of the spot holdings and r_F the futures contractual yield, or the yield to maturity of the deliverable at the terms of the contract. The parallel interest rate shift hypothesis can be written as $\Delta r_B = \Delta r_F = \Delta r$, and the ratio of price change variation can be calculated as :

$$h^* = - \frac{\Delta P}{\Delta r} \times \frac{1}{\frac{\Delta F}{\Delta r}} \tag{3}$$

An assumption is made with respect to the interest rate change Δr, for example 1%, and the induced price variation is calculated both for the spot and the futures contract based on the characteristics of the bonds.

Consider the following example. A portfolio of $1'000'000 face value of 30-year T-bonds paying 8% coupon is being hedged with T-bond futures. The bond's current yield to maturity r_B is 10% and the futures implied yield r_F is 10.2%. The bond pays coupon semi-annually and thus its price is 118.93 per $100 face value. The price at the level of 11% yield to maturity is calculated in a similar fashion and it is 108.72. The bond price change is ΔP = 108.72 - 118.93 = -10.21. The T-bond futures contract is quoted based on the price of a reference 8% coupon bond with twenty years to maturity. The futures prices for 10.2% and 11.2% are 81.38 and 74.66 respectively. Thus, the futures price change for one percent of rate change is ΔF = 74.66 - 81.38 = -6.72. It follows that the hedge ratio according to

(3) is equal to h* = -10.21/6.72 = -1.519. The hedge ratio, or the dollar size of the futures position per dollar hedged, is greater than one in this case because the hedged bond has a longer maturity than the deliverable instrument and thus a higher sensitivity to interest rate changes. The actual number of T-bond futures to be shorted is calculated by multiplying the hedge ratio with the face value of the spot position (in this case 1'000'000) and dividing by the face value of the deliverable under the contract (100'000). Thus the recommended futures positon using the yield sensitivity technique by direct calculation is short 15.19 T-bond futures. Alternatively, the same result is obtained by calculating the ratio of portfolio value change to the dollar change of one futures contract for one percent rate variation. (Bond prices are quoted in percentages of one percent of face value). The relevant calculations are summarized in table 1.

The direct calculation method of yield sensitivity has two limitations. First, by using the same changes in spot and futures implied rates, it assumes implicitly parallel interest rate shifts, and thus it ignores basis risk. Second, it is assumed that sensitivity varies proportional with the rate change. In reality the bond value as a function of rate is a convex function. The discount factors in the present value of cash flow model depend nonlinearly on the discount rate, and the value is a decreasing convex function of the rate. This makes the method dependent on the arbitrarily chosen rate variation Δr.

4. The Duration Approach

A more analytical model which adjusts for convexity by specifically taking into consideration the theoretical sensitivity to interest rate is the duration model. Duration represents the average time to repayment of a bond weighted by the present value of the cash payments. More precisely, if c_i represents the cash flow at time t_i, then duration is defined as:

$$D = \left(\sum_{i=1}^{n} \frac{c_i t_i}{(1+r)^i}\right) / \left(\sum_{i=1}^{n} \frac{c_i}{(1+r)^i}\right) \qquad (4)$$

where r is the appropriate discount rate; (e.g. if coupons are paid semi-annually, then r represents one half of the prevailing annual interest rate). It can be verified that the partial derivative of the bond price with respect to the interest rate r is $\delta P/\delta r$ = - DP, and thus duration measures the price elasticity of the bond, or the theoretical yield sensitivity. GAY, KOLB and CHIANG (1983) proposed an extension of the concept of duration to interest rate futures as the duration of the underlying deliverable instrument. Denote it by D_F and it is easy to verify that

$$\frac{\delta F}{\delta r} = - D_F F. \qquad (5)$$

Table 1: Calculations for the Yield Sensitivity Method.

Instrument	Current implied yield	Price at current yield	Price at yield + 1%	Value change ($)
30-year, 12% coupon bond ($1'000'000 face value)	10%	118.93	108.72	102'100
T-bond futures contract ($ 100'000 face value)	10.20%	81.38	74.66	6'720

Using the same rationale as in the technique involving direct calculation of yield sensitivity, it follows that

$$\frac{\delta P}{\delta F} = \left(\frac{\delta P}{\delta r}\right) / \left(\frac{\delta F}{\delta r}\right)$$

and thus the duration-based hedge ratio is:

$$h^* = - \frac{D_P P}{D_F F} \tag{6}$$

Duration is a linear measure of interest rate sensitivity, and approximation errors will occur for relatively large variations in interest rates. This inconvenience can be reduced by readjusting the futures position more frequently. The duration changes as a function of rate and instantaneously it correctly describes the convex dependency of bond price as a function of yield to maturity.

In the numerical example of section 3, the duration of the hedged bond is $D_P = 9.7574$, the duration of the T-bond futures contract is $D_F = 9.2974$, and thus the duration-based hedge ratio is

$$- \frac{9.7574 * 118.93}{9.2974 * 81.38} = -1.53.$$

While the price regression method focuses on the basis risk and fails to account for the changing of sensitivity to interest rates, the duration method takes into consideration the bond elasticity but fails to account for basis risk.

5. The Rate Diffusion Process Approach

For the practitioner hedging against interest rate risk is not a one-period task. The asset mix of the portfolio and the maturity changes with time, and thus the hedge ratio needs to be adjusted dynamically. Rollover of positions in the futures markets is also necessary because of the inavailability of contracts with settlement dates further into the future.

Assuming spot and futures interest rates to follow continuous time stochastic processes, GESKE and PIEPTEA (1987) proposed a continuously adjusted hedge ratio to control for risk and expected return. Consider a portfolio consisting of a number of bonds. All cash flows from the bonds comprising the portfolio can be aggregated and the portfolio can be thought of as a net bond. The spot interest rate state variable r_B is the yield to maturity of the resulting net bond. The futures implied rate r_F is the yield to maturity of the deliverable instrument based on the futures price. It is assumed that the two state variables, spot and futures implied rate follow correlated diffusion processes. Using stochastic calculus for differentials of compound functions of diffusion processes, GESKE and PIEPTEA (1987) derive the form of the diffusion process followed by the value of a mixed portfolio of spot and futures positions. Risk and return is controlled by adjusting the futures positions to attain target parameters of the portfolio value process. As this paper focuses on the methodological aspects of the hedging techniques, for ease of exposition the discrete time equivalent of the method is presented here.

Consider small discrete time intervals of length d(t) and estimate the variance of the rate changes over these intervals of the form $var(dr_B) = \sigma_B^2 dt$, $var(dr_F) = \sigma_F^2 dt$ and $cov(dr_B, dr_F) = \rho_{BF}\sigma_B\sigma_F dt$ using classical estimators. The correlation coefficient ρ_{BF} describes the correlation of the bond's yield to maturity with the futures contractual implied yield. This coefficient measures the hedging power of the futures contract relative to the fixed-income portfolio and it can be estimated using historical data. The minimum volatility hedge ratio is:

$$h_t^* = - \frac{D_B(t)P(t)\sigma_{BF}}{D_F(t)F(t)\sigma_F^2} = - \rho_{BF} \frac{D_B(t)P(t)\sigma_B}{D_F(t)F(t)\sigma_F} \tag{7}$$

for which the minimum volatility or variance of instantaneous return is

$$\sigma_V^2(h_t^*) = D_B^2\sigma_B^2(1-\rho_{BF}^2) \tag{8}$$

where D_B represents the duration of the bond, D_F represents the futures duration, P represents the portfolio market value, and F represents the invoice price of the deliverable instruments under the futures contract. The minus sign in relation (7) denotes that for hedging purposes, an opposite position must be taken in the futures market relative to the spot position. The length of the futures position is directly proportional to the hedging needs as measured by the product of the market value of the bond with its price elasticity, and inversely related to the hedging power of the futures contract as measured by the product of the futures price and futures duration. The position is also adjusted for the correlation factor ρ_{BF} which measures the relative hedg-ing effectiveness of the futures contract with respect to the spot position. When perfect correlation exists, ($\rho_{BF} = 1$), interest rate risk can be eliminat-ed.

6. Hedging With Multiple Futures

As formula (8) indicates, total volatility reduction is possible if the spot and futures implied rates are perfectly correlated. However, most often this is not the case, and basis risk is inherent in the unequal movement of the implied rates of return of the spot and futures positions. Cross hedging represents trading interest rate risk for basis risk. The actual amount by which variations of implied rates differ is maturity specific.

The basis risk can be partially diversified away by using multiple futures contracts whose contractual yields move in tandem with various segments of the yield curve. It is possible to decrease risk even further by taking advantage of natural hedges which occur across maturities included in the portfolio. The one "net bond" can be extended to a set of net bonds, each of them aggregating all the securities of a certain maturity. The cash market is characterized at all times by a set of yields for relevant maturities, which in fact represents an approximation of the yield curve. The futures market is characterized by a set of contractual implied yields.

Long term bonds are better correlated with futures contracts written on long-term instruments and the method accounts for the risk due to changes in the shape of term structure of interest rates. Also, by taking a global approach to portfolio hedging, the method takes advantage of synergy which becomes possible due to natural hedges and maturity diversification.

Let $P_1, P_2, \dots P_n$ be the market values of the (net) bonds comprising the spot portfolio and let $r_1^B, \cdots r_n^B$ be the corresponding yields to maturity. Similarly, denote by F_j and $r_1^F, \cdots r_m^F$ the invoice prices and respectively the contractual implied yields of the available futures. The state variables r^B and r^F follow diffusion processes with parameters of their own and their stochastic dependence is characterized by a variance-covariance matrix. By generalization of the GESKE/PIEPTEA (1987) result it can be shown that the minimum volatility position h_i in contract F_i is described by a vector $h^* = (h_1, \dots h_m)$ which is the solution to the system of linear equations $Ah = b$ where A is a m x m matrix, and b is a m-vector. Their elements are defined as:

$$a_{ij} = D_{F_i} D_{F_j} F_i F_j \sigma_{F_i} \sigma_{F_j} \rho_{F_i F_j}$$

$$b_j = - D_{F_j} F_j \sigma_{F_j} \sum_{i=1}^{n} D_{B_i} P_i \sigma_{B_i} \rho_{B_i F_j}$$

In matrix notation the minimum volatility hedge ratios are given by the vector

$$h^* = A^{-1} b. \qquad (9)$$

By algebraic manipulations, one can verify that in the particular case of n=m=1 the hedge ratio is consistent with the single futures contract method described in (7).

7. Empirical Results

In this section the effectiveness of hedging with multiple futures contracts is compared here with two single-contract strategies. Comparable hedging methods are used for the hedge of a long-term T-bond with a coupon of 7.85% annual interest payable semi-annually. The bond was issued in June 1960 and expired in May of 1986.

The following three rate-diffusion strategies are compared, using: 1) U.S. T-bill futures only, 2) T-bond futures contracts only, and 3) both T-bond and T-bill futures. The single-contract strategies (1 and 2), employ formula (7). In the case of one net bond and two futures (strategy 3), the minimum volatility ratios of (9) are:

$$h_1 = -\frac{D_B P \sigma_B}{D_{F_1} F_1 \sigma_{F_1}} \times \frac{(\rho_{BF_1} - \rho_{BF_1}\rho_{F_1F_2}}{(1-\rho_{F_1F_2})}$$

and

$$h_2 = \frac{D_B P \sigma_B}{D_{F_2} F_2 \sigma_{F_2}} \times \frac{(\rho_{BF_2} - \rho_{BF_2}\rho_{F_1F_2})}{(1-\rho^2_{F_1F_2})} \qquad (10)$$

A comparison of formulas (7) and (10) reveals that the hedge ratio for two futures can be derived for each position as if the contract was the only contract used, and by replacing the correlation coefficient with an adjustment factor which is a function of cross correlations of all the involved instruments.

Monthly bond price data covering the period 1/30/80 through 4/30/85 were provided by the Center for Research in Security Prices (CRSP), University of Chicago. The T-bond futures data were provided by the Chicago Board of Trade (CBOT). Trading on the CBOT, T-bond futures contracts are available for the delivery months March, June, September and December - and the quoted price is for a 8% coupon 20-year bond. At all times there are 8 contracts corresponding to settlement dates for the next two years. T-bill futures have the same delivery months as the T-bond futures and are traded on the International Money Market. The T-bill futures data were collected for the same period from the Wall Street Journal.

Dynamic hedging of a $100'000 face value portfolio consisting of the long-term bond is conducted by rebalancing the hedge ratio at the end of each month. This way a number of 62 monthly intervals are defined. Since no single futures contract coveres the entire period, it is necessary to roll over the positions in the futures markets. Price history of contracts must support both in-sample and out-of-sample parameter estimation. For this reason, the actual simulation covers periods 14-56 (02/27/81 - 30/09/84).

The parameters σ_B, σ_F and ρ_{BF} are estimated as follows. First, for a time variation dt = 1 month the implied spot position rate and futures rates variations are calculated. Classical inference techniques are used to derive the variance-covariance matrix of bond and futures implied rates of return. In real life the hedger has available historical information only and thus a realistic simulation must be based on out-of-sample parameter estimation. Thirteen historical data points are used to estimate the parameters required at each time. A reference benchmark is provided by comparing the results with similar techniques with in-sample data estimation.

Comparison of hedging effectiveness is conducted as suggested by EDERINGTON (1979) who evaluates variance reduction. The deviation from the promised yield to maturity is calculated and the variance of deviations is estimated. To compare the hedging performance for alternative maturities, the original time interval (months 14 - 56) is split into two parts, months 14-32 and months 32-56. The comparative study is conducted for the long period (14-56) and for the two subintervals. The hedging performance as measured by the reduction of variance of instantaneous return is indicated for each situation in table 2. The dynamic hedge with out-of-sample estimation with T-bond futures reduced variance of return deviations by 81% for the first subinterval (months 14-32) and by 65.20% for the second subinterval (months 32-56). The performance of the rate-diffusions method with out-of-sample estimation is almost as good as that of using in-sample estimation, which is an ideal situation but impossible in real life.

Overall better hedging results with single contracts have been obtained with T-bond futures than with

T-bill futures since given the maturity structure of the hedged bond, its yield to maturity displayed a better correlation with the contractual implied yield of the T-bond than that of the T-bill futures. This illustrates the usefulness of the coefficient ρ_{BF} as a measure of basis risk and instrument adequacy.
The multiple futures contract method produced systematically better results both for in- and out-of-sample estimation over all periods. As shown in column (5) of table 2 , the variance reduction for the multiple futures was between 71.65% and 82.05 % for out-of-sample parameter estimation, and between 83.09 and 90.39 for in-sample estimation. Table 2 reveals a substantial improvement of hedging effectiveness by using the multi-futures version of the rate diffusion method.

8. Concluding Remarks

This paper presents the applicability of recent theoretical results developed in the literature for hedging with interest rate futures. The more traditional methods such as variance minimization, regression and the yield sensitivity technique are briefly reviewed and the rate-diffusion method is explained in a practical framework. Its implication to hedging with multiple futures contracts is presented. It is shown how the method can be used to hedge against changes of both interest rates and shape of term structure of interest rates. Empirical analysis reveals that hedging with multiple futures is superior to other comparable hedging methods and the increase in sophistication is well justified.

Table 2: Hedging Effectiveness for Alternative Strategies.

		Variance Reduction (%)		
Method	Periods Covered	T-bond Futures	T-bill Futures	Multiple Futures
Rate diffusion;	14-56	78.73	68.69	81.98
	14-32	81.58	66.18	82.05
out-of-sample extimation	32-56	65.20	62.95	71.65
Rate diffusion;	14-56	83.87	80.90	88.76
	14-32	85.71	82.31	90.39
in-sample estimation	32-56	78.55	77.37	83.09

References

ANDERSON, R., and J. DANTHINE (1980): "Hedging and Joint Production", Journal of Finance 35, pp. 487-98.
EDERINGTON, L. (1979): "The Hedging Performance of The New Futures Markets",Journal of Finance 34, pp. 157-70.
GAY, G.D., R.W. KOLB and R. CHIANG (1983): "Interest Rate Hedging: An Empirical Test of Alternative Strategies", Journal of Financial Research 3, pp. 187-97.
GESKE, R.L., and D. PIEPTEA (1987): "Controlling Interest Rates Risk and Return with Futures",Review of Futures Markets 6, 1, pp. 64-86.
HILLIARD, J.E. (1984): "Hedging Interest Rate Risk With Futures Portfolios Under Term Structure Effects", Journal of Finance 34, pp. 1547-69.
JOHNSON, L. (1960): "The Theory of Hedging and Speculation in Commodity Futures", Review of Economic Studies 27, pp. 139-51.
MAKIN, J. (1978): "The Portfolio Theory and the Problem of Foreign Exchange Risk ", Journal of Finance 33, pp. 517-34.
McENALLY, R., and M. RICE (1979): "Hedging Possibilities In the Flotation of Debt Securities", Financial Management 8, pp. 12-18.
NELSON, R., and R. COLLINS (1985): "A Measure of Hedging Performance",Journal of Futures Markets 5, pp. 45-56.
STEIN, J. (1961): "The Simultaneous Determination of Spot and Futures Prices", American Economic Review 51, pp. 1012-25.
WORKING, H. (1953): "Futures Trading and Hedging", American Economic Review 48, pp. 314-43.